A TROPICAL INTERSECTION PRODUCT IN MATROIDAL FANS

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Abstract. We construct an intersection product on tropical cycles contained in the Bergman fan of a matroid. To do this we first establish a connection between the operations of deletion and restriction in matroid theory and tropical modifications as defined by Mikhalkin in [14]. This product generalises the product of Allermann and Rau [2], and Allermann [1] and also provides an alternative procedure for intersecting cycles which is not based on intersecting with Cartier divisors. Also, we simplify the definition in the case of one dimensional fan cycles in two dimensional matroidal fans and given an application of the intersection product to realisability questions in tropical geometry.

1. Introduction

One of the main goals of tropical geometry is to study classical algebraic geometry via polyhedral complexes. Tropicalisations of subvarieties of $(\mathbb{C}^*)^n$ are rational polyhedral complexes in $\mathbb{R}^n$ equipped with positive integer weights and satisfying the so-called balancing condition. For this reason tropical subvarities of $\mathbb{R}^n$ are considered to be polyhedral complexes with this added structure, [20], [14].

Before the advent of tropical geometry, Bergman fans were initially defined to be the *logarithmic limit sets* of complex algebraic varieties [4]. When equipped with appropriate weights they are tropical varieties in the above sense. For varieties defined by linear ideals, Sturmfels showed that the Bergman fan depends only on the underlying matroid. In addition, he generalised the Bergman fan construction to any loopless matroid [23]. Following this, an explicit construction of the fan involving matroid polytopes was given in [6], and its relation to the lattice of flats of the corresponding matroid has been studied in [3].

Bergman fans of matroids highlight the fact that not all tropical subvarieties have a classical counter-part. It is well-known that there exist matroids not representable over any field. However, in the tropics, as mentioned above every matroid has a geometric representation as a polyhedral fan. In this article, the Bergman fan of a matroid will also simply be called a matroidal fan.

Matroidal fans have many nice properties making them candidates for the local models of tropical non-singular spaces. Tropical linear spaces as studied by Speyer [22] and Speyer and Sturmfels [21] are locally matroidal fans. In addition, any codimension one cycle on a matroidal fan may be expressed as a tropical Cartier divisor, this is proved in Section 2.4. A particular case of this was proved by Allermann in [1] for so-called "tropical linear fans". These are skeleta of tropical hyperplanes, and not all matroidal fans arise in this way. However we show here that every matroidal fan of dimension can be obtained from $\mathbb{T}^n$ by a sequence of tropical modifications, see Subsection 2.4. A tropical modification can be thought of as a re-embedding of a tropical cycle. For this reason they are considered to be models of tropical affine space. The aim of this paper is to give a procedure for intersecting tropical cycles contained in matroidal fans which can be applied to more general smooth tropical spaces.

When the ambient space is $\mathbb{R}^n$, a tropical intersection product already exists, and is known as *stable intersection*, see [20], [14]. This intersection product is related to the fan displacement rule in toric intersection theory. In this case there are also various correspondence theorems relating the intersection of classical algebraic varieties in $(\mathbb{C}^*)^n$ to the stable intersection of their tropicalisations, see [17], [5]. The stable intersection of two tropical cycles $A, B \subset \mathbb{R}^n$ is supported on the skeleton

$(A \cap B)^{(k)}$ where k is the expected dimension of intersection. Moreover, the weights of the facets of the intersection are determined by the local structure of the complexes, see [10], [19].

One of the main differences and advantages of tropical stable intersection over classical theories is that products are defined on the level of cycles, even in the case of self-intersection. This greatly contrasts the situation in classical algebraic geometry, where some notion of equivalence is necessary in order to define an intersection product. A principal example of this is rational equivalence and Chow groups (see Chapter 1 of [7]).

On more general spaces Allermann and Rau [2] have defined intersections with tropical Cartier divisors, following a proposal of Mikhalkin in [14]. Moreover, they expressed the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$ as a product of tropical Cartier divisors, and using a procedure analogous to classical geometry may intersect any two cycles in $\mathbb{R}^n$; first by intersecting their Cartesian product with the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and then taking the pushforward of the result back to $\mathbb{R}^n$. It has been shown independently in [10], [19] that when the ambient space is $\mathbb{R}^n$ this intersection product coincides with the stable intersection mentioned above. Once again in Allermann and Rau's theory, the product is defined on the level of tropical cycles, there is no need to pass to equivalence classes.

The same phenomenon is true of the product on matroidal fans to be defined here. For a matroidal fan, $V \subset \mathbb{R}^n$ the product of two cycles $A, B \subset V$ is a well defined tropical cycle of the expected dimension contained in the fan V. As expected, the product is commutative, distributive and associative, see Proposition 3.9. The same proposition proves that the product is compatible with intersections of Cartier divisors from [2] and [14].

The method used here to construct the intersection product on cycles in a matroidal fan is similar in style to *moving lemmas* from classical algebraic geometry. This one approach to classical intersection theory begins with a notion of equivalence of cycles (such as rational equivalence), then given two cycles $X, Y \subset W$, one shows that there exists a class X' rationally equivalent to X which intersects Y properly. Naively speaking, many tropical cycles contained in a matroidal fan may not "move" on their own. In [14], there is an example of a rigid tropical cycle contained in a two dimensional matroidal fan in $\mathbb{R}^3$. This line and fan make an reappearance here in Figure 7 and Example 3.2. The idea is to construct a procedure which allows us to "split", instead of move, the tropical cycles into a sum in such a way that the intersection product on the components may be defined. The technique used here to construct this splitting comes from tropical modifications.

Tropical modifications, introduced by Mikhalkin in [14], are a simple yet powerful tool in tropical geometry. Working over a field $\mathbf{K}$, if $\mathbf{V} \subset \mathbf{K}^n$ is an algebraic variety and $\mathbf{f}$ a non-singular regular function on $\mathbf{V}$ with divisor $\mathbf{D} = \text{div}_{\mathbf{V}}(\mathbf{f})$, the graph of $\mathbf{f}$ gives an embedding of $\mathbf{V}$ in $\mathbf{K}^{n+1}$, with the image of $\mathbf{D}$ being contained in the hyperplane $\{z_{n+1} = 0\}$. This does not correspond to a very interesting operation classically, however performing the analogous procedure on tropical varieties produces a polyhedral complex with different topology. Often we work in $\mathbb{R}^n$ which is the tropicalisation of the torus $(\mathbf{K}^*)^n$. Performing the same procedure as above for a variety $\mathbf{V}$ in $(\mathbf{K}^*)^n$, the graph of a non-singular regular function $\mathbf{f}$ restricted to $\mathbf{V}$ gives an embedding of $\mathbf{V} \backslash \mathbf{D}$ to $(\mathbf{K}^*)^{n+1}$. Given a tropical variety $C \subset \mathbb{R}^n$ and a tropical function f on $\mathbb{R}^n$ the elementary open modification of C along f should be thought of simply as a reembedding of C with the divisor of f removed.

As mentioned previously, a k-dimensional matriodal fan in $\mathbb{R}^n$ may be obtained from $\mathbb{R}^k$ by a sequence of elementary open tropical modifications along functions with matroidal divisors. Given tropical cycles A, B in a matroidal fan $V \subset \mathbb{R}^n$ we may use this to express the product of two cycles $A, B \subset V$ as a sum of products in different matroidal fans V' and $D \times \mathbb{R}$, where V' is of lower codimension than V and D is of lower dimension. This procedure is repeated until the intersection is reduced to a sum of intersections in $\mathbb{R}^k$ where stable intersection may be applied.

The contents of the paper are as follows. Section 2 reviews the definitions of tropical cycles, regular and rational functions, tropical modifications/contractions and divisors in $\mathbb{R}^n$ from [14] and

[2]. Also we introduce their generalizations to $\mathbb{T}^n = [-\infty, \infty)^n$. In Subsection 2.4, the connection between tropical modifications and operations of matroid theory are established. Here, we work in tropical projective space which allows us to define the Bergman fan of a matroid with loops. An interesting discovery in this section is the need for **non-regular** modifications to produce matroidal fans, even in the case of realisable matroids, see Example 2.28.

In Section 3, tropical modifications and contractions are used to construct an intersection product on cycles in a matroidal fan. Again the idea is to split the cycles by using tropical modifications, and then to give the product on the components. Much of the work of this section is devoted to showing that this product is well-defined. In this section we also show that the product is associative, distributive, commutative and behaves as expected with divisors, Proposition 3.9.

Section 4 studies the case of one dimensional fan cycles in two dimensional matroidal fans. Firstly, Proposition 4.1 simplifies the intersection product in this case. Next if two one cycles contained in a two dimensional matroidal fan are also matroidal, Theorem 4.2 describes the intersection product of these cycles in terms of the lattice of flats of the corresponding matroids. Finally, using tropical modifications, Theorem 4.4 provides an obstruction to realising effective one dimensional tropical cycles in two dimensional fans by classical algebraic curves in planes. For instance, this shows that the tropical cycle B from Example 3.2 is not realisable. However, there are tropical curves in surfaces which are known to not be realisable but which are not obstructed by this theorem.

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2. Preliminaries

2.1. **Tropical cycles in $\mathbb{R}^n$.** Tropical cycles in $\mathbb{R}^n$ have been presented in various places, [2], [11], [14], [20]. We review the definitions here for completeness and to ease the generalisations to cycles in $\mathbb{T}^n$. First we give a summary of the necessary terminology.

A polyhedral complex P in $\mathbb{R}^n$ is a finite collection of polyhedra containing all the faces of its members and the intersection of any two polyhedra in P is a common face. We say a polyhedral complex P is **rational** if every face in P is defined by the intersection of half-spaces given by equations $\langle x, v\rangle \leq a$ where $a \in \mathbb{R}^n$ and $v \in \mathbb{Z}^n \subset \mathbb{R}^n$. The **support** $|P|$ of a complex is the union of all polyhedra in P as sets, and P is pure dimensional if $|P|$ is. A **facet** of P is a face of top dimension. Further, a polyhedral complex P is **weighted** if each facet F of P is equipped with a weight $w_F \in \mathbb{Z}$. A polyhedral complex P_1 is a **refinement** of a complex P_2 if their supports are equal and every face of P_2 is a face of P_1. For a complex P denote by $P^{(k)}$ the k-**skeleton** of P, meaning the union of all faces of P of dimension $i \leq k$.

Definition 2.1. *A pure dimensional weighted rational polyhedral complex $C \subset \mathbb{R}^n$ is balanced if it satisfies the following condition on every codimension one face $E \subset C$: Let $F_1, \dots F_s$ be the facets adjacent to E and v_i be a primitive integer vector such that for an $x \in E$, $x + \epsilon v_i \in F_i$ for some $\epsilon > 0$. Then,*

$$\sum_{i=1}^{s} w_{F_i} v_i,$$

is parallel to the face E, where w_{F_i} is the weight of the facet F_i, see the left hand side of Figure 1.

Definition 2.2. [14] *A tropical k-cycle $C \subset \mathbb{R}^n$ is a pure k-dimensional weighted, rational, polyhedral complex satisfying the balancing condition.*

A tropical k-cycle is **effective** if all of its facets have positive weights.

Definition 2.3. *Given two tropical cycles $A, C \subseteq \mathbb{R}^n$ we say A is a subcycle of C if $|A| \subseteq |C|$ and every open face of A is contained in a single open face of C.*

Remark If A is a subcycle of C, then there exists a refinement of the polyhedral structure on C so that A is a polyhedral subcomplex of C. Although we will not need to consider this refinement of C, the polyhedral structure on A as a subcycle of C will be important.

We can define an equivalence relation by declaring a cycle with all facets of weight zero to be equivalent to the empty polyhedral complex. The set of tropical k cycles in $\mathbb{R}^n$ modulo this equivalence will be denoted $Z_k(\mathbb{R}^n)$. This set forms a group under the operation of unions of complexes and addition of weight functions denoted by $+$. See [2], [14] for more details.

As mentioned in the introduction, there have been two approaches to intersections of cycles $\mathbb{R}^n$. Firstly, tropical stable intersection was defined for curves in $\mathbb{R}^2$ in [20] and for general cycles by Mikhalkin in [14]. The intersection product in $\mathbb{R}^n$ of Allermann and Rau is based on intersecting with the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$, see [2]. The two definitions have been shown to be equivalent in both [19], [10]. We review the definition of stable intersection in $\mathbb{R}^n$.

Definition 2.4. [20], [14] *Let $A \in Z_{m_1}(\mathbb{R}^n)$ and $B \in Z_{m_2}(\mathbb{R}^n)$, then their stable intersection, denoted $A.B$ is supported on the complex $(A \cap B)^{(k)}$ where $m = m_1 + m_2 - n$ with weights assigned on facets in the following way:*

(1) *If a facet $F \subset (A \cap B)^k$ is the intersection of top dimensional facets $D \subset A$ and $E \subset B$ and D and E intersect transversely, then*
$$w_{A.B}(F) = w_A(D)w_B(E)[\mathbb{Z}^n \colon \Lambda_D + \Lambda_E],$$
where Λ_D and Λ_E are the integer lattices parallel to the faces D and E respectively.

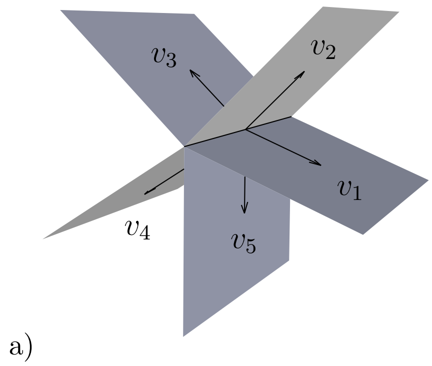
(2) *Otherwise for a generic vector v with non-rational projections and an $\epsilon > 0$, in a neighborhood of F, $A_\epsilon = A + \epsilon \cdot v$ and B will meet in a collection of facets $F_1 \dots F_s$ parallel to F such that the intersection at each F_i is as in the case (1) above. Then we set,*
$$w_F(A.B) = \sum_{i=1}^{s} w_{F_i}(A_\epsilon.B).$$

That the formula above is well-defined regardless of choice of the vector v follows from the balancing condition. In fact, the above weight calculation comes from the fan displacement rule for intersection of Minkowski weights from [8], for more details see [2] or [10]. By the equivalence of stable intersection and Allermann and Rau's intersection product on $\mathbb{R}^n$ shown in [19] [10], the following two propositions can be found in Section 9 of [2].

Corollary 2.5. [2] *Given A, B tropical cycles in $\mathbb{R}^n$ the following hold,*

(1) *$A.B \subset \mathbb{R}^n$ is a balanced tropical cycle.*
(2) $(A.B).C = A.(B.C)$
(3) $A.B = B.A$
(4) $A.(B + C) = A.B + A.C$

2.2. **Tropical cycles in $\mathbb{T}^n$.** The tropical numbers $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ form a semi-field equipped with the following operations:
$$\text{“}x \cdot y\text{”} = x + y \text{ and } \text{“}x + y\text{”} = \max\{x, y\}.$$
As the multiplicative and additive identity we have $1_\mathbb{T} = 0$, $0_\mathbb{T} = -\infty$ and tropical division corresponds to subtraction. We equip $\mathbb{T}^n = [-\infty, \infty)^n$ with the Euclidean topology, and will think of it as tropical affine n-space. It has a boundary which admits a natural stratification in the following way: Let $H_i = \{x \in \mathbb{T}^n \mid x_i = -\infty\}$, be the i^{th} coordinate hyperplane. Given a subset $I \subseteq [n] = \{1, \dots, n\}$ denote $H_I = \cap_{i \in I} H_i$, and
$$H_I^\times = \{x \in H_I | x \notin H_J \ I \subset J\}.$$



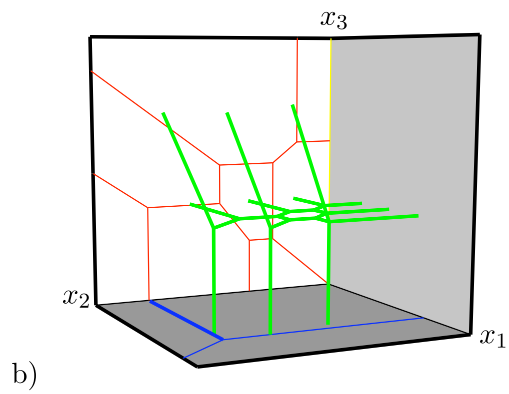


FIGURE 1. a) Balancing condition for a surface b) Cycles of sedentarity in $\mathbb{T}^3$.

Then,

$$\mathbb{T}^n = \coprod_{\emptyset \subseteq I \subseteq [n]} H_I^{\times}.$$

For every $I \in [n]$, we have $H_I = \mathbb{T}^{n-|I|}$ and $H_I^{\times} = \mathbb{R}^{n-|I|}$. Say a point $x \in H_I^{\times} \subset \mathbb{T}^n$ is of **sedentarity** I, this is denoted $S(x) = I$. The **order of sedentarity** of $x \in \mathbb{T}^n$ is the size of $S(x)$ and denoted $s(x)$.

We can generalise the definition of cycles in $\mathbb{R}^n$ to $\mathbb{T}^n$ by allowing cycles contained in the boundary strata of $\mathbb{T}^n$.

Definition 2.6. *A subset $B \subseteq \mathbb{T}^n$ is said to be of sedentarity I if it is the topological closure in $\mathbb{T}^n$ of some $B^o \subset H_I^{\times}$. A tropical k-cycle $C \subseteq \mathbb{T}^n$ of sedentarity I is the closure of a tropical k-cycle $C^o \subseteq H_I^{\times} = \mathbb{R}^{n-|I|}$, see the right hand side of Figure 1.*

Again let, $Z_{k,I}(\mathbb{T}^n)$ denote the quotient of the set of all k-cycles of sedentarity I by those with all zero weights. Given two cycles $A, B \in Z_{k,I}(\mathbb{T}^n)$ denote by $A + B$ the closure of $A^o + B^o$ as defined in $\mathbb{R}^{n-I}$. Then, $Z_{k,I}(\mathbb{T}^n) \cong Z_k(\mathbb{R}^{n-|I|})$ and we define,

$$Z_k(\mathbb{T}^n) = \bigoplus_{\emptyset \subseteq I \subseteq [n]} Z_{k,I}(\mathbb{T}^n).$$

Once again, a tropical cycle in $\mathbb{T}^n$ is **effective** if all of its facets are equipped with positive weights. Also, as in $\mathbb{R}^n$, a tropical cycle $A \subset \mathbb{T}^n$ is a **subcycle** of a cycle $C \subset \mathbb{T}^n$ if the supports satisfy $|A| \subseteq |C|$ and every face of A is contained in a face of C.

For cycles in $\mathbb{T}^n$ we define their intersection with a boundary hyperplane. Let $e_1, \dots e_n$ denote the standard basis of $\mathbb{R}^n$.

Definition 2.7. *Let $A \subseteq \mathbb{T}^n$ be a k-cycle of sedentarity I then*

- *if $i \in I$, set $A.H_i = \emptyset$.*
- *If $I = \emptyset$ then $A.H_i$ is supported on $(A \cap H_i)^{(k-1)}$ with the weight function defined as follows: Given a facet F of $(A \cap H_i)^{(k-1)}$ it is adjacent to some facets $\tilde{F}_1, \dots, \tilde{F}_s$ of A. Then,*

$$w_{A.H_i}(F) = \sum_{l=1}^{s} w_A(\tilde{F}_l)[\mathbb{Z}^n : \Lambda_{\tilde{F}_l} + \Lambda_i^{\perp}],$$

where $\Lambda_i^{\perp} = \{x \in \mathbb{Z}^n \mid \langle x, e_i \rangle = 0\}$.

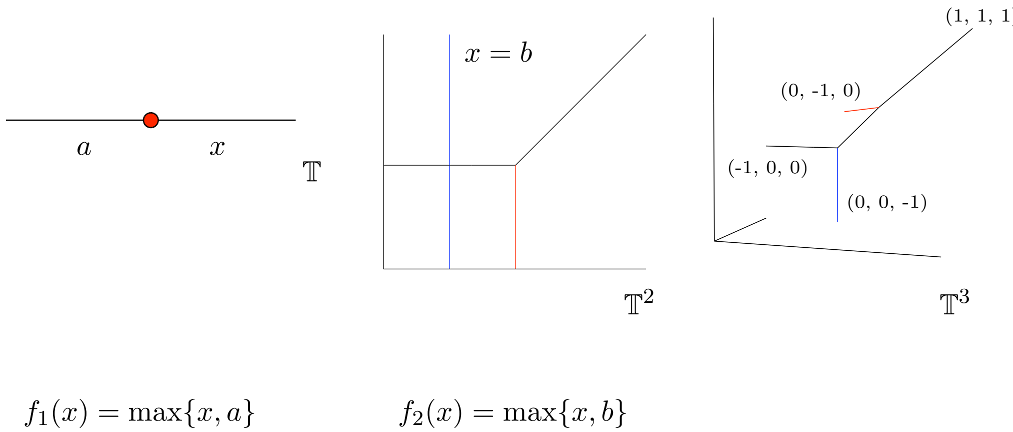


FIGURE 2. Two modifications of a tropical line

- *If $I \neq \emptyset$ and $i \notin I$ then $A.H_i$ is the intersection of $A.H_{i \cup I}$ calculated in $H_I = \mathbb{T}^{n-|I|}$ as in the case above.*

Every cycle $A \subseteq \mathbb{T}^n$ can be uniquely decomposed as a sum of its parts of different sedentarity and we extend the above definition to cycles of mixed sedentarity by linearity.

Proposition 2.8. *Given cycles $A, B \subseteq \mathbb{T}^n$ we have:*

(1) *$A.H_i$ is a balanced cycle.*
(2) *$(A + B).H_i = A.H_i + B.H_i$.*

Proof. For the balancing condition assume that A is of sedentarity $\emptyset$ and let $E \subseteq A.H_i$ be a face of codimension one which is in the interior of a face of $\mathbb{T}^n$ of sedentarity $\{i\}$. Let $\tilde{E}_j$ denote the faces of codimension one of A and of sedentarity $\emptyset$ which are adjacent to E. For $M >> 0$ let $L_M = \{x \in \mathbb{T}^n \mid x_i = -M\}$ then $\tilde{E}_j \cap L_M$ is in $A.L_M$ and $A.L_M$ is balanced at $\tilde{E}_j \cap L_M$. This means that

$$\sum_{\tilde{E}_j \subset \tilde{F}} w_{A.L_M}(\tilde{F} \cap L_M) v_{\tilde{F}} = \sum_{\tilde{E}_j \subset \tilde{F}} w_A(\tilde{F})[\mathbb{Z}^n : \Lambda_{\tilde{F}_l} + \Lambda_i^{\perp}] v_{\tilde{F}} = 0, \tag{1}$$

Let $\pi_I : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-|I|}$ be the linear projection with kernel $< e_i \mid i \in I >$. Then a facet $\tilde{F} \supset \tilde{E}_j$ is adjacent to a face $F \supset E$ if and only if $\pi_{I*}(v_{\tilde{F}}) = v_F$ where v_F is the primitive integer vector in $\mathbb{R}^{n-|I|}$ orthogonal to E generating F. Applying π_{I*} to (1) and taking the sum over all $\tilde{E}_j$ adjacent to E we obtain balancing at E.

When A and B are of equal sedentarity distributivity follows from the additivity of the weight function. For cycles of mixed sedentarity the intersection is defined by extending the product linearly, so the statement is trivial. This completes the proof. □

2.3. Tropical functions, modifications and divisors.

Definition 2.9. *Let U be an open subset of $\mathbb{T}^n$ and let $S(U) = \bigcup_{x \in U} S(x) \subset [n]$. A tropical regular function $f : U \longrightarrow \mathbb{T}$ is a tropical Laurent polynomial $f(x) =$ "$\sum_{\alpha \in \Delta} a_\alpha x^\alpha$" where $\Delta \subset \mathbb{Z}^n$ is such that for all $\alpha \in \Delta$, $\alpha_i \geq 0$ if $i \in S(U)$.*

A tropical regular function is a piecewise integral affine, convex function, whose graph is a finite polyhedral complex. Suppose $U \subseteq \mathbb{R}^n \subset \mathbb{T}^n$ then every regular function on U can be expressed as

a tropical Laurent polynomial $f(x) = \text{“}\sum_{\alpha \in \Delta} a_\alpha x^\alpha\text{”}$. If U contains a point x for which $x_i = -\infty$ for some i, then “$1/x_i$” $= -x_i = \infty \notin \mathbb{T}$. Distinct tropical polynomials may represent the same functions as some monomials may be redundant. Let $\mathcal{O}_{\mathbb{T}^n}(U)$ denote the semi-ring of regular functions on U and $\mathcal{O}_{\mathbb{T}^n}$ the regular functions on $\mathbb{T}^n$.

Tropical division corresponds to subtraction and so a rational function is of the form $h = \text{“}f/g\text{”} = f - g$ where $g \neq -\infty$. On $\mathbb{R}^n \subset \mathbb{T}^n$ such a function is always defined since it is the difference of two continuous functions. At the boundary of $\mathbb{T}^n$ where the function may take values $\pm\infty$ there may be a codimension two locus where the function is not defined. For example the function $f(x) = \text{“}\frac{x_1}{x_2}\text{”}$ on $\mathbb{T}^2$ at the point $(-\infty, -\infty)$. We denote the rational functions on $\mathbb{T}^n$ by $\mathcal{K}_{\mathbb{T}^n}$. Given a tropical cycle $C \subseteq \mathbb{T}^n$ (or $C \subseteq \mathbb{R}^n$) regular functions and rational functions on C, denoted $\mathcal{O}_C$ and $\mathcal{K}_C$ respectively, are obtained by restriction of $\mathcal{O}_{\mathbb{T}^n}$ and $\mathcal{K}_{\mathbb{T}^n}$ (or $\mathcal{O}_{\mathbb{R}^n}$ and $\mathcal{K}_{\mathbb{R}^n}$) .

Given a cycle $C \subseteq \mathbb{T}^n$ we may consider the graph $\Gamma_f(C) \subset \mathbb{T}^{n+1}$ of a function $f \in \mathcal{O}_C$ restricted to C. The graph $\Gamma_f(C)$ is still a rational polyhedral complex, and it inherits weights from C. Since f is piecewise affine $\Gamma_f(C)$ is not necessarily balanced. At any unbalanced codimension one face E of $\Gamma_f(C)$ we may attach the closed facet F_E, generated by E and the direction $-e_{n+1}$, more precisely,

$$F_E = \{(x, c) \mid x \in \overline{E}, c \in (x, -\infty]\}.$$

Moreover, there exists a unique integer weight on F_E such that the resulting complex is now balanced at E. Let the **undergraph** of $\Gamma_f(C)$ be the weighted rational polyhedral complex

$$\mathcal{U}(\Gamma_f(C)) = \bigcup_{\substack{E \subset \Gamma_f(C) \\ codim(E)=1}} F_E,$$

with weights described above. Finally, the weighted complex

$$\tilde{C} = \Gamma_f(C) \cup \mathcal{U}(\Gamma_f(C)),$$

is a tropical cycle. Let $\delta : \mathbb{T}^{n+1} \longrightarrow \mathbb{T}^n$ be the linear projection with kernel generated by e_{n+1}.

Definition 2.10. *Given acycle $C \subseteq \mathbb{T}^n$ and a regular function $f \in \mathcal{O}_C$, the regular elementary modification of C along the function f is*

$$\delta : \tilde{C} \longrightarrow C.$$

Often the term “elementary modification” will be used to denote only the cycle $\tilde{C}$, in this case the existence of the map $\delta : \tilde{C} \longrightarrow C$ and function f is implied. The cycle C will also sometimes be referred to as the contraction of $\tilde{C}$. This notation is similar in style to that used for blow-ups in classical algebraic geometry.

A **regular modification**, respectively **regular contraction**, is any composition of regular elementary modifications, respectively contractions. Using modifications we define the divisor of a function on a cycle $C \subseteq \mathbb{T}^n$.

Definition 2.11. *Let $f, g : \mathbb{T}^n \longrightarrow \mathbb{T}$ be regular functions and suppose $g \neq 0_{\mathbb{T}}$ and let $C \subset \mathbb{T}^n$ be a cycle and $\delta_f : \tilde{C} \longrightarrow C$ the elementary modification of C along the function f. Then,*

(1) *$div_C(f) = \delta_f(\tilde{C}.H_{n+1})$*
(2) *If $h = \text{“}f/g\text{”}$ then $div_C(h) = div_C(f) - div_C(g)$.*

Given a regular elementary modification $\delta : \tilde{C} \longrightarrow C$ along a function f, we say that $\text{div}_C(f)$ is the **divisor of the modification**.

All of the above definitions given for cycles in $\mathbb{T}^n$ restrict to cycles in $\mathbb{R}^n$, and we may modify cycles in $\mathbb{R}^n$ along regular functions in $\mathcal{O}_{\mathbb{R}^n}$, not just functions on $\mathbb{T}^n$. In particular we have $\text{div}_{C^o}(f) = \text{div}_C(f) \cap \mathbb{R}^n$. This definition coincides with the definition of divisors from [2] and

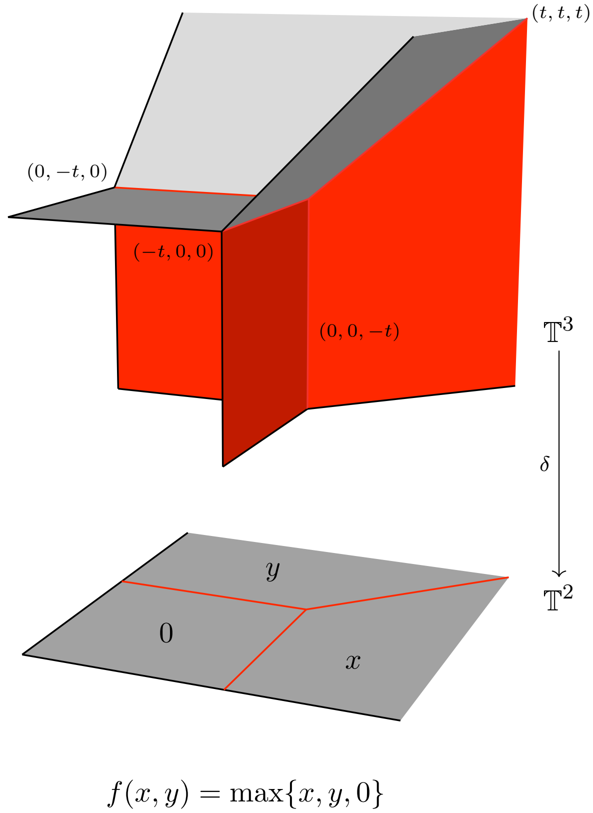


FIGURE 3. A modification P of the tropical affine plane $\mathbb{T}^2$

[14]. From the definition of divisors we notice that a tropical invertible function on $\mathbb{T}^n$ is tropical multiplication by a scalar $x \in \mathbb{T}^\times = \mathbb{R}$ (so addition), and a tropical invertible function on $\mathbb{R}^n$ is a Laurent monomial.

Proposition 2.12. [2], [10] *Given tropical rational functions $f, g \in \mathcal{K}_{\mathbb{R}^n}$ and tropical cycles $A, B \subset \mathbb{R}^n$*

(1) $div(f)_{\mathbb{R}^n}.A = div_A(f)$
(2) $div_{A+B}(f) = div_A(f) + div_B(f)$
(3) $div_A(\text{“}f \cdot g\text{”}) = div_A(f) + div_A(g)$

The following proposition will be needed later on in Proposition 2.16.

Proposition 2.13. *For functions $f, g \in \mathcal{K}_{\mathbb{T}^n}$ and cycles $A, B \subset \mathbb{T}^n$. We have,*

$$div_{A+B}(f) = div_A(f) + div_B(f).$$

Proof. This follows directly from Proposition 2.8 and Definition 2.11. $\square$

Following part (1) of Proposition 2.12 we have:

Corollary 2.14. *If $f \in \mathcal{O}(\mathbb{R}^n)$ then $div_C(f)$ is effective for every effective cycle $C \subset \mathbb{R}^n$.*

Regular modifications can be generalized to rational functions with effective divisors on effective cycles C. The construction of the modification $\tilde{C}$ is the same as in the regular case. The resulting cycle $\tilde{C}$ is effective since the weights of the facets of $\mathcal{U}(\Gamma_f(C))$ correspond to the weights of the facets of $\text{div}_C(f)$, which is assumed to be effective.

Definition 2.15. *Given an effective cycle $C \subset \mathbb{T}^n$ and a rational function $f \in \mathcal{K}_C$ such that $div_C(f)$ is effective, the elementary modification of C along the function f is*

$$\delta : \tilde{C} \longrightarrow C,$$

where again $\tilde{C} = \Gamma(C) \cup \mathcal{U}(\Gamma_f(C))$.

A **modification**, respectively **contraction**, is any composition of elementary modifications, respectively contractions. For an elementary modification of a cycle in $\mathbb{T}^n$ we can define pullback and pushforward maps on subcycles.

Definition 2.16. *Let $C \subset \mathbb{T}^n$ be an effective cycle, $f \in \mathcal{K}_C$ be a function with effective divisor on C and $\delta : \tilde{C} \to C$ be the elementary modification along f. We define the following:*

(1) *The pushforward map of cycles, $\delta_* : Z_k(\tilde{C}) \to Z_k(C)$ is given by $\delta_* A = \delta(A)$ with weight function,*

$$w_{\delta_* A}(F) = \sum_{F_i \subset A, \delta(F_i) = F} w_A(F_i)[\bar{\Lambda}_{F_i} : \Lambda_F],$$

where $\bar{\Lambda}_{F_i}$ is the image under δ of the integer lattice generated by F_i and Λ_F is the integer lattice generated by F.

(2) *The pullback map of cycles $\delta^* : Z_k(C) \to Z_k(\tilde{C})$. For a cycle $A \in Z_k(C)$, $\delta^* A$ is the modification of A along the function f.*

Clearly the cycle $\delta^* A$ is contained in $\tilde{C}$. Notice that $\delta_* \delta^* A = A$ but $\delta^* \delta_* A$ is not always equal to A. Also, the pullback of an effective cycle may not be effective if the modification of C is given by a rational function, an example of this can be found in [2] and [14]. Moreover, since the definition of the weight function on the pushforward is additive δ_* is a homomorphism. The pullback map is also a group homomorphism by Proposition 2.13.

2.4. **Bergman fans of matroids and tropical modifications.** Here we study tropical modifications in relation to Bergman fans of matroids. This section provides a correspondence between tropical modifications and existing constructions in matroid theory. There are many equivalent ways of describing a matroid, here we will most often use the rank function. So we write a matroid as $M = (E, r)$ where $E = \{0, \dots N\}$ is the ground set and r is a rank function, $r : \mathcal{P}(E) \longrightarrow \mathbb{N} \cup \{0\}$, satisfying certain axioms, see [18]. The **flats** of a matroid are the subsets $F \subset E$ such that the rank function satisfies $r(F) < r(F \cup i)$ for all $i \in E$ not contained in F. By convention, the ground set E is also a flat. The flats of a matroid M form a lattice, which we will denote Λ_M.

In this section the focus is on the following basic concepts from matroid theory and their connections to tropical modifications. We include their definitions for the reader not familiar with matroid theory. Again, for a comprehensive introduction to the subject see [18].

Definition 2.17. *Let $M = (E, r)$ be a matroid where $E = \{0, \dots, n\}$ and $e \in E$, then*

(1) *The deletion with respect to i, $M \backslash i$ is the matroid $(E \backslash i, r|_{E \backslash i})$.*
(2) *The restriction with respect to i, M/e is the matroid $(E \backslash i, r')$ where $r'(I) = r(I \cup i) - r(i)$.*
(3) *A matroid Q is a single element quotient of M if there exists a matroid N on a ground set $E' = E \cup i'$ such that $N \backslash i' = M$ and $N/i' = Q$, and N is called a single element extension of M.*

Deletions and restrictions can be performed with respect to a subset $I \subset E$, these will be denoted $M \backslash I$ and M/I respectively. Also a matroid Q will be called a quotient of M if there is a matroid N with ground set $E \cup F$ such that $N \backslash F = M$ and $N/F = Q$, and N will simply be called an extension.

We wish to consider a projective version of the Bergman fan of a matroid M contained in tropical projective space.

Definition 2.18. [14] *Tropical projective space is*

$$\mathbb{TP}^n = (\mathbb{T}^{n+1} \setminus (-\infty, \dots, -\infty))/(x_0, \dots, x_n) \sim (x_0 + \lambda, \dots, x_n + \lambda)$$

for $\lambda \in \mathbb{R}$.

Tropical projective space is topologically the n-simplex. We can equip $\mathbb{TP}^n$ with tropical homogeneous coordinates $[x_0 : \cdots : x_n]$ similarly to the classical setting. It may be covered by $n+1$ charts $U_i = \{[x_0 : \cdots : x_n] \mid x_i = 0\} = \mathbb{T}^n$. Moreover, the boundary of $\mathbb{TP}^n$ is stratified; given $\emptyset \neq I \subseteq \{0, \dots, n\}$ we have a face of the n-simplex corresponding to the subset of $\mathbb{TP}^n$ where $x_i = -\infty$ in homogeneous coordinates. Moreover such a face is isomorphic to $\mathbb{TP}^{n-|I|}$. Similar to $\mathbb{T}^n$, a tropical cycle A in $\mathbb{TP}^n$ is the closure of a cycle in $\mathbb{R}^{n-|I|} \subset \mathbb{TP}^{n-|I|}$ identified as one of the boundary strata of $\mathbb{TP}^n$.

We now review of the construction of the Bergman fan of M, denoted $B(M)$, in terms of the lattice of flats from [3]. Recall that a **loop** of a matroid is an element $i \in E$ that is not contained in any basis. First assume that M is loopless, meaning it contains no loops. For $1 \leq i \leq n$ set $v_i = -e_i$ and $v_0 = \sum_{i=1}^n e_i$, where $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n \subset \mathbb{T}^n$. Let Λ_M denote the collection of flats of M. For every chain $\emptyset \neq F_1 \subset \cdots \subset F_k \neq E$ in Λ_M we have a corresponding k dimensional cone in $B(M)$ given by the positive span of $v_{F_1}, \dots, v_{F_k}$ where $v_{F_l} = \sum_{i \in F_l} v_i$. Finally, $B(M)$ is the closure in $\mathbb{TP}^n$ of the union of all such polyhedral cones. This is the **fine** polyhedral structure on $B(M)$ as defined in [3]. This construction is a projectivisation of the definitions given in [6], [3] up to a reflection caused by the use of the *max* convention instead of *min*.

Example 2.19. *A geometric example of matroids is to consider a hyperplane arrangement* $\mathcal{A} = \cup_{i=0}^n L_i$ *in* $\mathbb{CP}^k$. *We can define a matroid on* $E = \{0, \dots, n\}$, *by the rank function* $r(E') = codim(\cap_{i \in E'} L_i)$. *Suppose the rank of the associated matroid is* $k+1$ *which is equivalent to* $\cap_{i=0}^n L_i = \emptyset$. *Each hyperplane is given by a linear form* f_i. *Using these forms we can define a map:*

$$\begin{aligned} \phi : \mathbb{CP}^k &\longrightarrow \mathbb{CP}^n \\ x &\mapsto [f_0(x) : \cdots : f_n(x)] \end{aligned}$$

If M *is loopless then* $\phi(\mathbb{CP}^k) \cap (\mathbb{C}^*)^n = \mathbb{CP}^k \setminus \mathcal{A}$, *the complement of the hyperplane arrangement. The logarithmic limit set of* $\phi(\mathbb{CP}^k) \cap (\mathbb{C}^*)^n$ *is the Bergman fan of the associated matroid* $M_{\mathcal{A}}$, *see* [4], [23] *for more details.*

Again if $M_{\mathcal{A}}$ *is the matroid arising from a hyperplane arrangement* $\mathcal{A}$ *we can interpret the above operations geometrically. The deletion,* $M_{\mathcal{A}} \backslash i$, *corresponds to the arrangement given by removing the* i^{th} *hyperplane,*

$$\mathcal{A}' = \mathcal{A} \backslash L_i,$$

and the restriction $M_{\mathcal{A}}/i$ *is the arrangement on* $\mathbb{CP}^{n-1}$ *obtained by restricting the arrangement* $\mathcal{A}$ *to* L_i.

$$\mathcal{A}'' = \cup_{j \neq i}(L_i \cap L_j).$$

For more on this see Section 1 *of* [16].

If i *is a loop then* $codim(i) = 0$ *meaning* L_i *is the degenerate hyperplane defined by the linear form* $f_i = 0$, *and so* $\phi(\mathbb{CP}^k)$ *is contained in the* i^{th} *coordinate hyperplane of* $\mathbb{CP}^n$. *From this we next define the Bergman fan in* $\mathbb{TP}^n$ *for a matroid with loops.*

Definition 2.20. *Given a matroid* $M = (E, r)$, *let* $I \subset E$ *denote its collection of loops. Then the complex* $B(M)$ *is contained in the boundary of* $\mathbb{TP}^n$ *corresponding to* $x_l = -\infty$ *for all* $l \in I$ *and is equal to* $B(M \backslash I) \subseteq \mathbb{TP}^{n-|I|}$.

By the following lemma all quotients of a matroid M can be represented geometrically as Bergman fans of matroids which are polyhedral subcomplexes of $B(M)$. Here we use the fine polyhedral subdivision from [3] as described in this section.

Lemma 2.21. *A matroid Q is a quotient of M if and only if $B(Q) \subseteq \mathbb{TP}^n$ is a polyhedral subcomplex of $B(M) \subseteq \mathbb{TP}^n$.*

Proof. We may assume M is loopless and that Q is a single element quotient of M, since every quotient can be formed by a sequence of single element quotients. Moreover, by Proposition 7.3.6 of [18] Q is a quotient of M if and only if $\Lambda_Q \subseteq \Lambda_M$. So supposing Q is loopless, the lemma follows immediately from the above statement and the construction of $B(M)$ in terms of the lattice of flats. If Q contains loops $L \subset E$, then $B(Q)$ is contained in the boundary stratum of $\mathbb{TP}^n$ corresponding to $x_l = -\infty$ for all $l \in I$. A face of $B(Q)$ corresponding to a chain of flats $I = F_0 \subset F_1 \subset \cdots \subset F_s \neq \{0, \dots, n\}$ of Λ_Q, is contained in the boundary of $B(M)$ if and only if the same chain is a chain in Λ_M and the lemma is proved. □

The next proposition relates tropical modifications, contractions and divisors to matroid extensions, deletions and restrictions, respectively. Recall that an element $i \in E$ is a **coloop** of a matroid $M = (E, r)$ if i is contained in every basis of M, i.e. $i \in B$ for every $B \subset E$ for which $r(B) = |B| = r(E)$. If a matroid M contains m coloops then the corresponding Bergman fan $B(M) \subset \mathbb{TP}^n$ contains an m dimensional subspace of $\mathbb{R}^n$.

Proposition 2.22. *Let M be a rank $k+1$ matroid on the ground set $E = \{0, \dots, n\}$. Suppose $i \in E$ is neither a loop nor a co-loop, then in every chart $U_j = \{x \in \mathbb{TP}^n \mid x_j \neq -\infty\} = \mathbb{T}^n \subset \mathbb{TP}^n$ there is an elementary tropical modification*

$$\delta_j : B(M) \cap U_j \longrightarrow B(M \backslash i) \cap U_j$$

with corresponding divisor $B(M/i) \cap U_j$.

Proof. For the lattice of flats of deletions and restrictions we have:

$$\Lambda_{M\backslash i} = \{F \subseteq E\backslash i \mid F \text{ or } F \cup i \text{ is a flat of M}\}$$
$$\Lambda_{M/i} = \{F \subseteq E\backslash i \mid F \cup i \text{ is a flat of M}\}.$$

Let $\delta_i : \mathbb{T}^n \longrightarrow \mathbb{T}^{n-1}$ be the projection in the direction of e_i. Then the image under δ_i of a k-dimensional cone of $B(M) \cap U_j$ corresponding to a chain of flats $F_1 \subset \cdots \subset F_k$ is still a k dimensional cone if and only if $i \notin F_k$. In other words, if and only if the corresponding chain is a chain of flats of $\Lambda_{M\backslash i}$. Therefore, we have

$$\delta(B(M) \cap U_j) = B(M\backslash i) \cap U'_j,$$

where U'_j is a chart of $\mathbb{T}^{n-1}$. In addition, δ contracts a k-dimensional face of $B(M) \cap U_j$ if and only if $i \in F_k$. Thus the image of all contracted faces is exactly $B(M/i) \cap U'_j \subset B(M\backslash i) \cap U'_j$.

By the next lemma the codimension one cycle $B(M/i) \cap U'_j$ must be the divisor of a tropical rational function f on $B(M\backslash i) \cap U'_j$. Then up to tropical multiplication by a constant (addition) this function must satisfy

$$\Gamma_f(B(M\backslash i) \cap U'_j) \subset B(M) \cap U_j$$

and so it must be the function of the modification δ. □

Lemma 2.23. *Let $B(M) \subset \mathbb{TP}^n$ be the Bergman fan of a matroid, and $V = B(M) \cap U_i \subset \mathbb{T}^n$ for some $i \in \{0, \dots, n\}$. If $D \subset V$ is a codimension one tropical subcycle then there exists a tropical rational function $f \in \mathcal{K}_{\mathbb{T}^n}$ such that $div_V(f) = D$.*

Proof. First suppose $V = \mathbb{T}^n$, and that D has order of sedentarity 0, then the statement is equivalent to showing that every codimension one cycle in $\mathbb{R}^n$ is the divisor of a tropical function $f \in \mathcal{K}_{\mathbb{R}^n}$. If D is effective, it is a tropical hypersurface and is given by a tropical polynomial by [20]. When D is not effective the following argument is due to an idea of Anders Jensen. Let D^- denote the

collection of facets of D which have negative weights. For a face E in D^-, there exists a $v \in \mathbb{Z}^n$ and $a \in \mathbb{R}$ such that $< x, v >= a$ for all $x \in E$. Define a regular function $h_E : \mathbb{T}^n \longrightarrow \mathbb{T}$, by

$$h_E(x) = \max\{0, -w_E(< x, v > -a)\}$$

where $w_E < 0$ is the weight of E in D. The function h_E is given by the tropical polynomial = "$ax^{-w_E v} + 1_{\mathbb{T}}$", and $\text{div}_{\mathbb{T}^n}(h_E)$ is an affine hyperplane containing E and equipped with positive weight $-w_E$. Let $h : \mathbb{T}^n \longrightarrow \mathbb{T}$ be given by $h(x) = \sum_{E \in D^-} h_E(x)$, this corresponds to the tropical product of the tropical polynomials, so h is again a tropical polynomial. Moreover, $D + \text{div}_{\mathbb{T}^n}(h)$ is an effective cycle of order of sedentarity zero in $\mathbb{T}^n$, and thus is the divisor of a tropical polynomial f. By part (3) of Proposition 2.12, $D = \text{div}_{\mathbb{T}^n}(f - h)$, and the difference $f - h$ is a tropical rational function.

If D does not have order of sedentarity 0 then D contains boundary hyperplanes of $\mathbb{T}^n$ corresponding to $x_i = -\infty$ for a choice of i, equipped with an integer weight. This boundary cycle is the divisor of the Laurent monomial x_i^w. Because D is of codimension one, it decomposes into a sum of cycles of order of sedentarity 0 or 1 so we are done.

Now if the fan $V \subset \mathbb{T}^n$ is the closure of a k-dimensional subspace in $\mathbb{R}^n$ and $D \subset V$ a codimension one cycle. There is a unique surjective linear projection $\delta : V \longrightarrow \mathbb{T}^k$ with kernel generated by standard basis directions. The image $\delta(D) \subset \mathbb{T}^k$ is isomorphic to D as an integral polyhedral complex. Moreover equipped with the weights from D, $\delta(D)$ is a balanced codimension one cycle in $\mathbb{T}^k$. Therefore, it is the divisor of a tropical rational function f on $\mathbb{T}^k$. Let $\tilde{f}$ be the pullback of this function to $\mathbb{T}^n$. It is again a tropical rational function and we have $\text{div}_V(\tilde{f}) = D$.

For the general case, take a linear projection $\delta : V \longrightarrow V'$, with kernel generated by e_i. Denote the divisor of δ by $D' \subset V'$. We may assume by induction that a codimension one cycle in V' (and similarly, $D' \times \mathbb{R}$) is the divisor of a tropical rational function. Recall the pushforward and pullback maps defined for an elementary modification in Definition 2.16. Then the cycle $\delta_* D \subset V'$ is the divisor of a tropical rational function f and the cycle $\delta^* \delta_* D$ is the divisor of the pullback $\tilde{f} = f(\delta)$. The difference $\delta^* \delta_* D - D$ is a cycle contained in the undergraph of δ and may be considered as a cycle in $D' \times \mathbb{R}$ (for details see Lemma 3.5 of the next section). So, $\delta^* \delta_* D - D$ is the divisor of some tropical rational function g on $D' \times \mathbb{R}$. Moreover, we may choose g so that $D = \text{div}_V(\tilde{f} - g)$, so the claim is proved. □

The above lemma shows that the tropical analogues of Weil divisors and Cartier divisors on the Bergman fan of a matroid are equivalent. However, effective tropical codimension one cycles are not always given by regular tropical functions. Examples of this appear in [14] and [2] and also in Example 2.28 at the end of this section.

We remark that even when i is a loop the above proposition holds, but in a particular sense where the function on $B(M\backslash i)$ producing the modification is the constant function $f = -\infty$. The divisor of such a function is all of $B(M\backslash i)$ which is equal to $B(M/i)$, if i is a loop.

A **basis** of a matroid $M = (E, r)$ is a subset $B \subseteq E$ such that $|B| = r(B) = r(E)$.

Corollary 2.24. *Given a k-dimensional Bergman fan $B(M) \subseteq \mathbb{TP}^n$, every contraction $\delta : B(M) \longrightarrow \mathbb{TP}^k$ corresponds to a choice of basis of M.*

Proof. Given a basis B of M the deletion $M\backslash B^c$ produces the uniform matroid $U_{k+1,k+1}$ corresponding to $\mathbb{TP}^k$. If we delete along a set which is not the complement of some basis then we decrease the rank of the matroid, meaning at some step we deleted a coloop. This does not correspond to a tropical contraction. □

From now on the focus will be on matroidal fans and matroidal contraction charts. To simplify the notation we will drop the use of $B(M)$ and just insist that a fan is matroidal, we will only recall the underlying matroid when necessary.

Definition 2.25. *We call a fan $V \subset \mathbb{T}^n$ (or $\mathbb{R}^n$) matroidal if there exists a matroid M on $n+1$ elements such that $V = B(M) \cap U_i$ for some coordinate chart $U_i = \{x_i \neq -\infty\} \subset \mathbb{TP}^n$, (or $V = B(M) \cap \mathbb{R}^n$).*

In the next section we will be concerned only with matroidal fans in $\mathbb{R}^n$. For this we make clear the notion of **open matroidal tropical modifications**.

Definition 2.26. *Let $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^{n-1}$ be matroidal fans. An elementary open matroidal tropical modification is a modification*

$$\delta : \tilde{V} \longrightarrow V,$$

where $V \subset \mathbb{R}^n$, and $\tilde{V} \subset \mathbb{R}^{n+1}$ are matroidal and the divisor $D \subset V'$ of the modification is also matroidal or empty.

As mentioned in the introduction, an open elementary matroidal modification $\delta : \tilde{V} \longrightarrow V$ along a function f should be thought of as a embedding of V with the divisor $\mathrm{div}_V(f)$ removed. As before, an open matroidal modification is a composition of elementary matriodal open modifications.

Let $V \subset \mathbb{R}^n$ be a k-dimensional Bergman fan of a matroid M. Let $\mathbf{K}$ be the field of Puiseux series with coefficients in a field $\mathbf{k}$ of characteristic p. We say V is **realisable** over a field of characteristic p if there exists a k-plane $\mathbf{V} \subset (\mathbf{K}^*)^n$ such that $\mathrm{Trop}(\mathbf{V}) = V$, (see for example [12] for definitions of Trop of an algebraic variety). This is equivalent to the corresponding matroid M being realisable in characteristic p, see [21].

Proposition 2.27. *Let $V_M \subset \mathbb{R}^k$ be a k-dimensional matroidal fan corresponding to the matroid M. If V_M is obtained from $\mathbb{R}^k$ by a sequence of elementary regular matroidal modifications then V_M is realisable over a field of characteristic zero.*

Proof. Let $\delta : V \longrightarrow V'$ be the first elementary open regular matroidal modification of a sequence, and let $D \subset V'$ be the corresponding divisor. Then V' corresponds to the matroid $M \backslash i$ for some $i \in E$ the ground set of M. Also D corresponds to the matroid M/i. By induction we may assume that V' is realisable over a field of characteristic zero. Without loss of generality we may also suppose that $D \neq \emptyset$ and that there is a regular tropical function f with $\mathrm{div}_{V'}(f) = D$ and such that $\mathrm{div}_{\mathbb{R}^{n-1}}(f) = V_f$ is matroidal. Then the cycle V_f defined by f, is also realisable in the above sense. By Theorem 4.3 from [22], the tropical stable intersection of two matroidal is always realisable over the field of Puiseux series with coefficients in $\mathbb{C}$. So the modification of V' with center D is realisable by the graph of the function giving $\mathbf{D}$ restricted to $\mathbf{V}'$, where $\mathbf{D}$ and $\mathbf{V}'$ realize D and V' respectively. ☐

Remark Bergman fans obtained via modification by regular functions *do not* correspond to regular matroids, where regular means being realisable over every field. For instance the matroid $U_{2,4}$ which is not realisable over the field $\mathbb{F}_2$ corresponds to the four valent tropical line in $\mathbb{TP}^3$ which can be obtained by modifications along regular functions.

Modification along regular functions is sufficient to ensure realisability, but it is by no means necessary.

Example 2.28. *The embedding of the moduli space of tropical rational curves with 5 marked points, $\mathcal{M}_{0,5}^{trop}$ into $\mathbb{R}^5$ (see [15], [21], [9]) is the first example of a realisable fan not obtained by a sequence of modifications along regular functions. The rays of a fine polyhedral subdivision of $\mathcal{M}_{0,5}^{trop}$ may be labelled by distinct pairs $\{i,j\} \subset \{1,\dots,5\}$ as in Figure 4. See [15], [3] for more details. It was shown in [3] that $\mathcal{M}_{0,n}^{trop}$ corresponds to the Bergman fan of the complete graphical matroid*

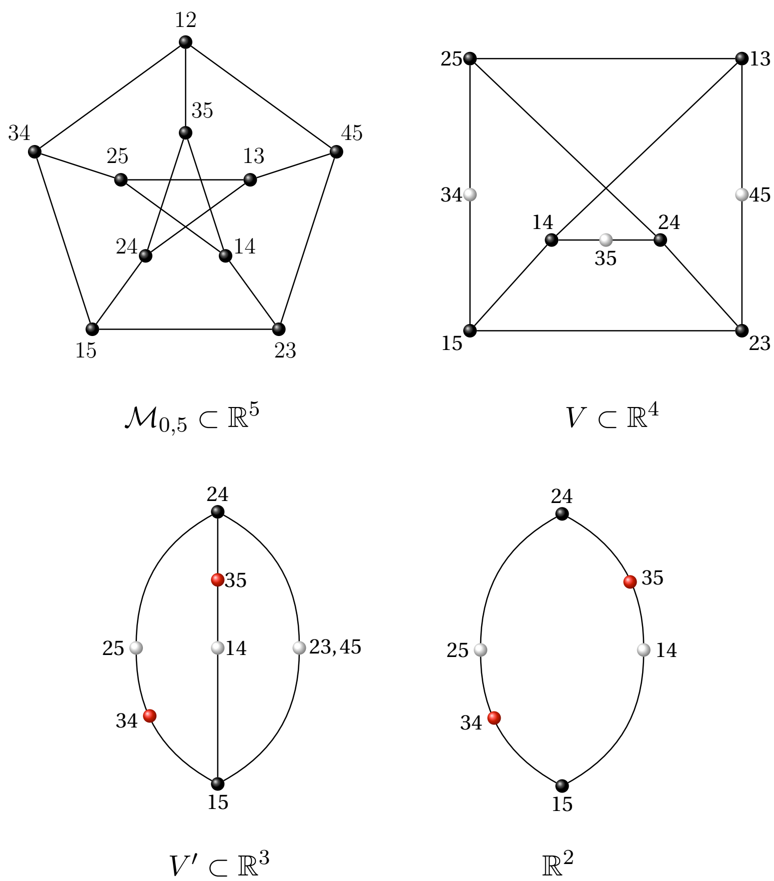


FIGURE 4. The link about the origin (or Bergman complex [23]) of the sequences of modifications producing $\mathcal{M}_{0,5}^{trop}$. The divisor at each step is marked in white, the ij indicate the cone of $\mathcal{M}_{0,5}^{trop}$ corresponding to a combinatorial type of curve, see [15], [9] .

K_{n-1}. Tropical contractions of $\mathcal{M}_{0,n}^{trop}$ correspond to the deletion of an edge of K_{n-1}. Therefore, the very first elementary tropical contraction of $\mathcal{M}_{0,5}^{trop}$ is unique by the symmetry of K_4. The link of singularity of the fans obtained by a series of elementary contractions starting from $\mathcal{M}_{0,5}^{trop}$ and finishing at $\mathbb{R}^2$ are drawn in Figure 4. The divisors of each modification marked in white. It will be shown in Section 4 that the corresponding divisor of this contraction cannot be the divisor of a regular function on $\mathbb{R}^4$ restricted to V by showing that its tropical self intersection is not effective, which would contradict Corollary 2.14.

For open matroidal modifications we have the following proposition regarding the pullback and pushforward cycle maps given in Definition 2.16.

Proposition 2.29. *Given an open matroidal modification $\delta : \tilde{V} \longrightarrow V$ the maps $\delta^* : Z_k(V) \longrightarrow Z_k(\tilde{V})$ and $\delta_* : Z_k(\tilde{V}) \longrightarrow Z_k(V)$ are group homomorphisms for all k, and $\delta_*\delta^* = id$.*

Proof. It was already mentioned that the pushforward and pullback maps are group homomorphisms when the modification is elementary. Therefore, we must only show that the maps δ_*, δ^* are well defined when we compose open elementary modifications. Suppose $\delta : \tilde{V} \longrightarrow V$ is the composition of two open matroidal modifications. Set $\delta_2 : \tilde{V} \longrightarrow V_2$ and $\tilde{\delta}_1 : V_2 \longrightarrow V$, so that $\tilde{\delta}_1\delta_2 = \delta$, and denote the other sequence of modifications by $\delta_1 : \tilde{V} \longrightarrow V_1$ and $\tilde{\delta}_2 : V_1 \longrightarrow V$,

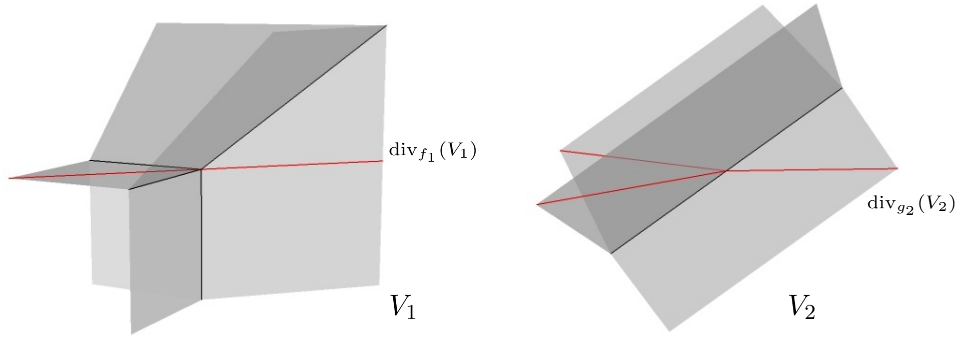


FIGURE 5. The two cycles V_1, $V_2 \subset \mathbb{R}^3$ from Example 2.30, the divisors $\mathrm{div}_{f_1}(V_1) \subset V_1$, $\mathrm{div}_{g_2}(V_2) \subset V_2$ are drawn in red in each case.

so that $\tilde{\delta}_2\delta_1 = \delta$, (see Example 2.30 for a case when the fans V_1 and V_2 differ). Without loss of generality we may suppose the kernels of $\delta_1, \delta_2 : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+1}$ are generated by e_{n+1} and e_{n+2}, respectively. Then the maps $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are linear projections $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$, with kernels e_{n+1} and e_{n+2} respectively.

For the pushforwards, the sets satisfy, $\tilde{\delta}_i\delta_j(A) = \tilde{\delta}_j\delta_i(A)$, since the δ_i's and $\tilde{\delta}_i$'s are orthogonal projections. Let $\mathcal{C}$ denote the closure of the collection of facets of C contracted by both δ_1, δ_2. If a facet F of A is outside of $\mathcal{C}$ then its contribution to the weight of $\delta(F) \subset \delta_*A$ is the same if we permute the order of contractions. So assume $F \subset \mathcal{C}$, then the lattice index may be rewritten as, $[\delta_{i*}\Lambda_F : \Lambda_{\delta_i(F)}] = [\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp}]$ and F contributes a weight of,

$$w_A(F)[\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp}][\mathbb{Z}^n : \Lambda_{\delta_1(F)} + \Lambda_j^{\perp}] = w_A(F)[\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp} \cap \Lambda_j^{\perp}],$$

to $\delta(F)$. Which is independent of the order of contractions.

For the pullbacks, take a cycle A in V and let $\Gamma(A) \subset \mathbb{R}^{n+2}$ denote the graph of A along either pair of functions yielding the modification. Although the pairs of functions may differ, (see Example 2.30), the resulting graphs must be the same. Let $\tilde{A}$ denote the pullback of A along the composition $\tilde{\delta}_2\delta_1$ and $\tilde{A}'$ denote the pullback of A along the composition $\tilde{\delta}_1\delta_2$. Since $\tilde{A}$ and $\tilde{A}'$ are modifications of cycles in V the restriction of the linear projections δ_1 and δ_2 to $\tilde{A}$ and $\tilde{A}'$ are either one to one or send a half line to a point.

If E_A is an unbalanced codimension one face of $\Gamma(A)$, then it is unbalanced only in the e_{n+1} and e_{n+2} directions. First, if $\tilde{V}$ contains one or both of the faces:

$$\{x - te_{n+1} \mid x \in E_A \text{ and } t \in \mathbb{R}_{\geq 0}\},\ \{x - te_{n+2} \mid x \in E_A \text{ and } t \in \mathbb{R}_{\geq 0}\},$$

then these are the only facets of $\tilde{A}$ adjacent to E_A and not contained in $\Gamma(A)$, and similarly for $\tilde{A}'$. The balancing condition at E_A guarantees that the weights are the same, (remark that if $\Gamma(A)$ is already balanced in one of these directions then we do not need to add the corresponding facet).

For an unbalanced codimension one face E_A of $\Gamma(A)$ suppose the above faces do not exist. Then there is a single facet $F_{\tilde{A}}$ of $\tilde{A}$ adjacent to E_A and not in $\Gamma(A)$. Otherwise the projections δ_1 and δ_2 restricted to $\tilde{A}$ would have a finite fiber of size at least two. The same holds for $\tilde{A}'$, whose single face satisfying these conditions we call $F_{\tilde{A}'}$. Now $\tilde{A} - \tilde{A}'$ must be balanced at E_A and so the faces $F_{\tilde{A}}$ and $F_{\tilde{A}'}$ are the same and equipped with the same weights.

In this case there may be codimension one faces of $F_{\tilde{A}}$ at which there are other facets of $\tilde{A}$ adjacent. This occurs when the divisor $D_1 \subset V_1$ of the modification δ_1 is contained in the undergraph of the modification $\tilde{\delta}_2$ and $\tilde{\delta}_2^*A$ and in addition intersects $D_1 \subset V_1$ in some codimension one face. Call the resulting codimension one face G_A of $F_{\tilde{A}} \subset \tilde{A}$. Then G_A is contained in the skeleton of $\tilde{V}$ and it is also a face of $F'_{\tilde{A}} \subset \tilde{A}'$. If the cycles are unbalanced at G_A the other facets adjacent to it

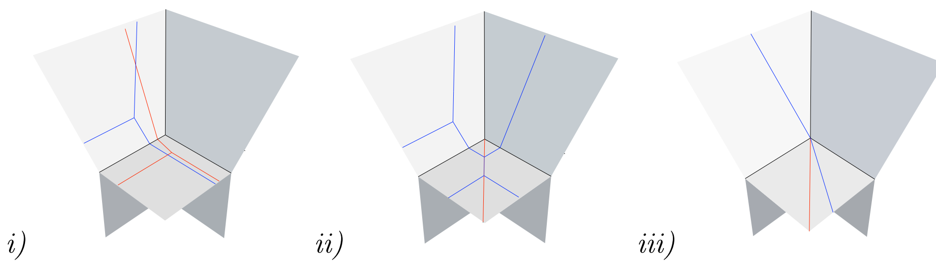


FIGURE 6. Cycles in the standard hyperplane in $\mathbb{R}^3$. i) Transverse intersection ii) Weakly transfer intersection iii) Neither

in $\tilde{A}$ and $\tilde{A}'$ must be:

$$\{x - te_{n+1} \mid x \in G_A \text{ and } t \in \mathbb{R}_{\geq 0}\}, \ \{x - te_{n+2} \mid x \in G_A \text{ and } t \in \mathbb{R}_{\geq 0}\},$$

otherwise the projections δ_1, δ_2 would have a finite fiber of size greater than one. Again, by the balancing condition the weights of these faces in $\tilde{A}$ and $\tilde{A}'$ agree. □

The following example shows a composition of open matroidal modifications for which the intermediary fans V_1, V_2 appearing in the proof above are not the same.

Example 2.30. *Consider the fan $V \subset \mathbb{R}^4$ obtained from $\mathbb{R}^2$ via two elementary open modifications, δ_1, δ_2. The first modification is along the function $f_2(x, y) = \max\{x, y, 0\}$ and yields the cycle $V_1 \subset \mathbb{R}^3$ shown on the left of Figure 5. The next modification is taken along the function $f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by*

$$f_1(x, y, z) = \max\{x, y\} + \max\{z, 0\} - \max\{x, y, z, 0\}.$$

It may be verified that the following different sequence of modifications yields the same fan, $V \subset \mathbb{R}^4$, after a change of coordinates. If one first modifies $\mathbb{R}^2$ along the function $g_1(x, y) = \max\{x, y\}$, to obtain a cycle $C_2 \subset \mathbb{R}^3$, see the right hand side of Figure 5. Next, modify V_2 along the function $g_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, given by $g_2(x, y, z) = \max\{z, 0\}$. Notice on the one hand V is produced by a composition of two elementary regular modifications, and on the other by an elementary regular modification composed with an elementary modification along a rational function.

3. INTERSECTIONS IN MATROIDAL FANS

In this section we intersect tropical subcycles of an open matroidal fan $V \subset \mathbb{R}^n$, so throughout we restrict our attention to open matroidal tropical modifications. Set $\dim(V) = k, \dim(A) = m_1, \dim(B) = m_2$, and the expected dimension of intersection of A and B to be $m = m_1 + m_2 - k$. Also for any complex C whose support is contained in V, let $C^{(s)}$ denote the s-dimensional skeleton of C with respect to the refinement induced by the inclusion to V.

Definition 3.1. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and $A, B \subset V$ be subcycles.*

1. *$A \cap B$ is proper in V if $A \cap B$ is of pure dimension m or is empty.*
2. *$A \cap B$ is weakly transverse in V if every facet of $(A \cap B)^{(m)}$ is in the interior of a facet of V.*
3. *$A \cap B$ is transverse in V if it is proper, weakly transverse and every facet of $A \cap B$ comes from facets of A and B intersecting transversely.*

Example 3.2. *The standard hyperplane $P \subset \mathbb{R}^3$ was shown in Figure 3, it is obtained by modifying $\mathbb{R}^2$ along the standard tropical line. Let A be the sub-cycle parameterized by $(t, t, 0)$ and B be the union of the positive span of the rays $(0, 1, 1)$, $(1 - d, -d, 0)$, $(d - 1, d - 1, -1)$, see Figure 7.*

The curves A and B intersect only at the vertex p of the fan. This intersection is proper but not weakly transverse. Moreover both cycles are rigid in P, meaning they cannot be moved in P by

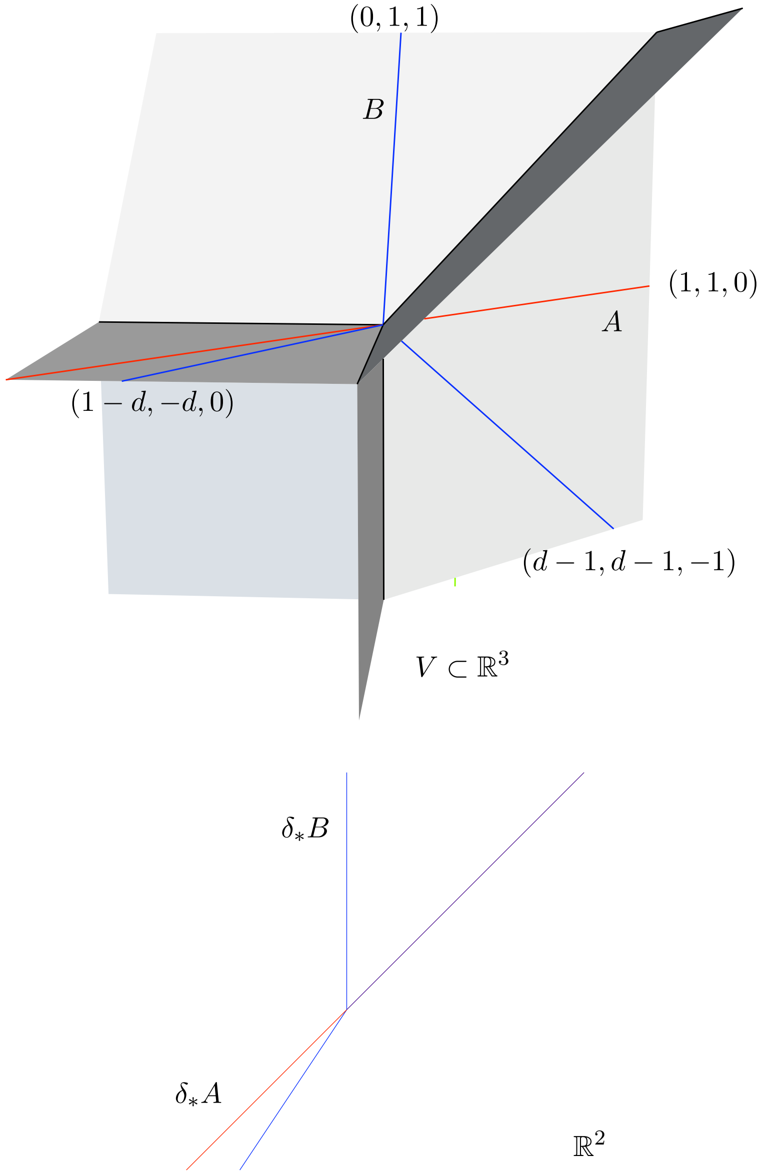


FIGURE 7. Tropical cycles in the standard hyperplane in $\mathbb{R}^3$ along with the image under the contraction to $\mathbb{R}^2$.

a translation. Consider the contraction $\delta : P \longrightarrow \mathbb{R}^2$, given by projecting in the e_3 direction. Set $\Delta_A = \delta^*\delta_*A - A$ and $\Delta_B = \delta^*\delta_*B - B$. An intersection product should of course be distributive, so we ought to have,

$$\begin{aligned} A.B &= (\delta^*\delta_*A - \Delta_A).(\delta^*\delta_*B - \Delta_B) \\ &= \delta^*\delta_*A.\delta^*\delta_*B - \delta^*\delta_*A.\Delta_B - \Delta_A.\delta^*\delta_*B + \Delta_A.\Delta_B \end{aligned}$$

Now, the cycles $\delta^*\delta_*A, \delta^*\delta_*B$ are free to move in P in the same way that δ_*A, δ_*B are free to move in $\mathbb{R}^2$. By translating $\delta^*\delta_*A, \delta^*\delta_*B$ until they intersect transversally and then translating back we can associate the weight,

$$w_{\delta^*\delta_*A.\delta^*\delta_*B}(p) = 1 = w_{\delta_*A.\delta_*B}(\delta(p)).$$

The cycles Δ_A, Δ_B are contained in the undergraph of the modification, see Figure 8, and are free to move in this direction. Also the cycle $\delta^*\delta_* A$ restricted to the undergraph is just $div_A(f) \times \mathbb{R}$, and similarly for $\delta^*\delta_* B$.

Now the cycles Δ_A, Δ_B may be moved by a translation into a single facet of P, see Figure 8. We can calculate

$$w_{\delta^*\delta_* A.\Delta_B}(p) = w_{\Delta_A.\delta^*\delta_* B}(p) = 0,$$

and

$$w_{\Delta_A.\Delta_B}(p) = 1 - d.$$

Combining all of these we obtain:

$$\begin{aligned} w_{A.B}(p) =& w_{\delta^*\delta_* A.\delta^*\delta_* B}(p) - w_{\delta^*\delta_* A.\Delta_B}(p) \\ & - w_{\Delta_A.\delta^*\delta_* B}(p) + w_{\Delta_A.\Delta_B}(p) = -d + 2. \end{aligned}$$

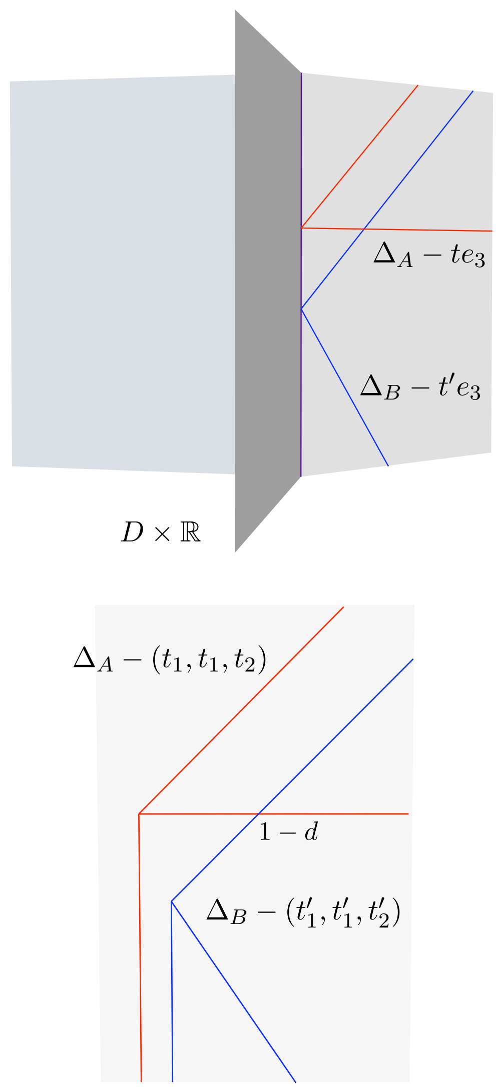


FIGURE 8. The second and then third translations of the cycles Δ_A, Δ_B from Example 3.2.

Our aim is to obtain a general procedure to split cycles contained in a matroidal fan V in a way so that they may be intersected. To do this we first need some technical definitions and lemmas.

Definition 3.3. *Given $\delta : V \longrightarrow V'$ an elementary open matroidal modification, let f denote the corresponding tropical rational function and D its divisor. Let $A \subset V$ a be cycle, then denote:*

(1) $\Delta_A = \delta^*\delta_*A - A$.
(2) $D_A = div_{\delta_*A}(f) \times \mathbb{R} \subset D \times \mathbb{R}$.

Lemma 3.4. *Given $\delta : V \longrightarrow V'$ an elementary open matroidal modification, let f denote the corresponding tropical rational function and D its divisor. Let A, B be subcycles of V then,*

(1) $\Delta_{A+B} = \Delta_A + \Delta_B$
(2) $D_{A+B} = D_A + D_B$.

Proof. The first statement is clear since δ^*, δ_* are homomorphisms, and the second follows from $\mathrm{div}_{A+B}(f) = \mathrm{div}_A(f) + \mathrm{div}_B(f)$. □

Lemma 3.5. *Let $\delta : V \longrightarrow V'$ be an elementary open matroidal modification along the rational function f and having divisor D. If a cycle $A \subset V$ is in Ker δ_*, then it is contained in the closure of the undergraph $\mathcal{U}(\Gamma_f(V'))$. In particular, it is also a subcycle of $D \times \mathbb{R}$ where $\mathbb{R}$ is the affine space spanned by the kernel of δ.*

Proof. Away from the divisor $D \subset V$ the map δ is one to one thus no cancellation of facets can occur in δ_*A outside of D. So $\delta(A)$ must be contained in D which implies the lemma. □

A quick check shows that Δ_A is in the kernel of δ_*, since $\delta_*\delta^* = \mathrm{id}$. Therefore, for an elementary open modification of matroidal fans $\delta : V \longrightarrow V'$ and any cycle $A \subset V$ we have $\Delta_A, D_A \subset D \times \mathbb{R}$, where $D \subset V'$ is the divisor of the modification. Using this we define an intersection product on V in terms of a product on V' and $D \times \mathbb{R}$.

Definition 3.6. *Given cycles $A, B \subset V \subset \mathbb{R}^n$ and an elementary open matroidal modification $\delta : V \longrightarrow V'$ with associated divisor D, define,*

$$A.B = \delta^*(\delta_*A.\delta_*B) + C_{A.B}$$

with

$$C_{A.B} = \Delta_A.\Delta_B - \Delta_A.D_B - D_A.\Delta_B,$$

where these products are calculated in the matroidal fan $D \times \mathbb{R} \subset \mathbb{R}^n$.

The above definition gives the product of two cycles A, B in V as a sum of products of cycles in fans V' and $D \times \mathbb{R}$, one of which is of lower codimension, and the other containing the linear space spanned by the kernel of δ. Continuing to apply this procedure to V' and D we continue to decrease the codimension or increase the dimension of the affine linear space contained in the fan and we can eventually reduce the intersection product in V to a sum of pullbacks of stable intersections in $\mathbb{R}^k$, where k is the dimension of V. A priori this definition depends on the choice of all contraction charts. Before showing the above definition is independent of the chosen charts in Proposition 3.11 we state some properties of the intersection product as defined relative to a fixed collection of open matroidal contractions.

Lemma 3.7. *Suppose $\delta : V \longrightarrow V'$ is an elementary open modification of matroidal fans and A, B are cycles in V'. The intersection product in V from Definition 3.6 calculated via the modification δ satisfies*

$$\delta^*A.\delta^*B = \delta^*(A.B).$$

Proof. In this case $\Delta_A, \Delta_B = 0$ so the term $C_{A,B}$ from Definition 3.6 is also 0. □

Corollary 3.8. *Suppose the matriodal fan $V \subset \mathbb{R}^n$ is a k-dimensional subspace of $\mathbb{R}^n$, and let $\delta : V \longrightarrow \mathbb{R}^k$ be an open matroidal contraction. For subcycles A, B in V we have,*

$$A.B = \delta^*(\delta_* A.\delta_* B).$$

Proposition 3.9. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and A, B, C be subcycles of V. Then the intersection product given in Definition 3.6 relative to any choice of contraction charts satisfies the following:*

(1) *$A.B$ is a balanced cycle contained in V*
(2) *$A.C = C.A$*
(3) *$A_1.(A_2 + A_3) = A_1.A_2 + A_1.A_3$*
(4) *$A_1.(A_2.A_3) = (A_1.A_2).A_3$*
(5) *$div_A(g) = div_V(g).A$*

Proof. The above properties all follow by induction. The base case being $V = \mathbb{R}^k$, where all of the above properties are satisfied. Suppose we have chosen, $\delta : V \longrightarrow V'$ as the first elementary open matroidal contraction, and let its divisor be $D \subset V'$. We may assume all of the properties stated above hold for intersections in V' and $D \times \mathbb{R}$.

For (1), the weighted balanced complex, $A.B$ is the sum of $\delta^*(\delta_* A.\delta_* B)$ and $C_{A.B}$ which are both balanced by the induction assumption, so it is balanced. Commutativity also follows immediately by induction. By Lemma 3.4 and distributivity for products in V' and $D \times \mathbb{R}$, we get distributivity in V.

For associativity, first notice that

$$\Delta_{A_i.A_j} = \Delta_{A_i}.D_{A_j} + D_{A_i}.\Delta_{A_j} - \Delta_{A_i}.\Delta_{A_j} \tag{2}$$
$$D_{A_i.A_j} = D_{A_i}.D_{A_j}. \tag{3}$$

The first line follows from the definition of $\Delta_{A_i.A_j}$. The statement (3) follows from Lemma 3.10 which follows this proposition. Then,

$$A_1.(A_2.A_3) = \delta^*(\delta_* A_1.(\delta_* A_2.\delta_* A_3)) - \Delta_{A_1}.D_{A_2.A_3} - D_{A_1}.\Delta_{A_2.A_3} + \Delta_{A_1}.\Delta_{A_2.A_3}$$

Assuming associativity in V and $D \times \mathbb{R}$ and using commutativity we can remove brackets and write:

$$\begin{aligned} A_1.(A_2.A_3) =& \delta^*(\delta_* A_1.\delta_* A_2.\delta_* A_3) + \\ & \sum_{\substack{1 \leq i < j \leq 3 \\ k \neq i,j}} \Delta_{A_i}.\Delta_{A_j}.D_{A_k} - D_{A_i}.D_{A_j}.\Delta_{A_k} - \Delta_{A_1}.\Delta_{A_2}.\Delta_{A_3} \end{aligned}$$

Regrouping terms and using (2) and (3) we get,

$$\begin{aligned} A_1.(A_2.A_3) &= \delta^*((\delta_* A_1.\delta_* A_2).\delta_* A_3) - \Delta_{A_1.A_2}.D_{A_3} - D_{A_1.A_2}.\Delta_{A_3} + \Delta_{A_1.A_2}.\Delta_{A_3} \\ &= (A_1.A_2).A_3. \end{aligned}$$

Lastly, given a divisor $D = \text{div}_V(g)$ we may write it as $\delta^*\delta_* D - \Delta_D$. Then $\tilde{g}(x) = g(\delta(x))$, is the function of the divisor $\delta^*\delta_* D$ where f is the function of the modification δ. So $\tilde{g} - g$ gives Δ_D by part 3 of Proposition 2.12. The result follows by distributivity and by applying the induction hypothesis to both parts. □

We require a final lemma before proving that the product is independent of the choice of contractions.

Lemma 3.10. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and A, B be subcycles of V, set*

$$\tilde{A} = A \times \mathbb{R}, \quad \tilde{B} = B \times \mathbb{R}, \quad \text{and} \quad \tilde{V} = V \times \mathbb{R}.$$

Then, we may choose contraction charts so that by Definition 3.6 we have

$$\tilde{A}.\tilde{B} = A.B \times \mathbb{R} \subset \tilde{V}.$$

Proof. The above statement holds for stable intersections in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. If V corresponds to a matroid M on E then $\tilde{V}$ corresponds to a matroid $\tilde{M}$ on $E \cup e$ with bases $B \cup e$ for every base B of M, in other words we have added a coloop e to the matroid M. Given an elementary open modification of matroidal fans, $\delta : V \longrightarrow V'$ with divisor D we have a corresponding elementary open modification $\tilde{\delta} : \tilde{V} \longrightarrow \tilde{V}'$ with divisor $\tilde{D} = D \times \mathbb{R}$ and $\tilde{V}' = V' \times \mathbb{R}$. In order to define the product in $A.B$, a collection of contractions are fixed. To intersect $\tilde{A}, \tilde{B}$ in $\tilde{V}$, simply choose the corresponding collection of contractions of $\tilde{V}$. Applying Definition 3.6 we obtain the lemma by induction. $\square$

Theorem 3.11. *The intersection product from Definition 3.6 is independent of the choice of open matroidal contractions.*

Proof. Fix a matroidal fan $V \subset \mathbb{R}^n$ and subcycles A, B of V. We may assume by induction that the product is well-defined on $D \times \mathbb{R}$ and V' where $\delta : V \longrightarrow V'$ is any elementary open matroidal modification and D is its associated divisor.

By Corollary 2.24 any two open matroidal contractions $\delta, \delta' : V \longrightarrow \mathbb{R}^k$ can be related by a series of basis exchanges. So it suffices to check two things: that we may transpose the order of any two elementary open contractions to $\mathbb{R}^k$ and obtain the same intersection cycle and that if $\delta : V \longrightarrow V'$ is the composition of any two elementary open matroidal modifications, we may permute the order of the elementary contractions and obtain the same product. In other words we must show that the definition does not depend on the paths taken in the following two diagrams:

$$\begin{tikzcd} & V \arrow[dl, "\delta_1"'] \arrow[dr, "\delta_2"] & \\ V_1 \arrow[dr, "\tilde{\delta}_2"'] & & V_2 \arrow[dl, "\tilde{\delta}_1"] \\ & V' & \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "\delta_1"'] \arrow[dr, "\delta_2"] & \\ \mathbb{R}^k \arrow[rr, dotted, "e_2 \mapsto e_1"'] & & \mathbb{R}^k \end{tikzcd}$$

We will start by showing the latter, let $\delta_1, \delta_2 : V \longrightarrow \mathbb{R}^k$ be two elementary open matroidal contractions. Then V is of codimension one in $\mathbb{R}^{k+1}$ and thus corresponds to a corank one matriod M. Suppose without loss of generality that the open contractions δ_i correspond to the deletion of the element i from the corresponding matroid, Then we may assume that $i = 1, 2$ are not coloops of M. If we exchange any two non coloop elements i and j of a corank one matroid M we obtain a matroid isomorphism. Also, restricting the matroid M to i or j produces isomorphic matroids $M/i \cong M/j$. Therefore the divisors $D_i, D_j \subset \mathbb{R}^k$ of the corresponding elementary open matroidal modifications $\delta_i, \delta_j : V \longrightarrow \mathbb{R}^k$ can be identified as well as the functions f_i, f_j on $\mathbb{R}^k$.

First we will construct cycles $\tilde{A}, \tilde{B} \subset V$ such that $\delta_{1*}\tilde{A} = \delta_{2*}\tilde{A} \subset \mathbb{R}^k$ and $\delta_i^*\delta_{i*}\tilde{A} = \tilde{A}$ for $i = 1, 2$ and similarly for $\tilde{B}$. Then by the above remarks concerning the two modifications the definition of the product $\tilde{A}.\tilde{B} = \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B})$ does not depend on the choice of $i = 1, 2$.

To construct $\tilde{A}$ and $\tilde{B}$, let $\mathcal{C} \subset V$ denote the union of all faces of V that are not generated by the vectors v_1, v_2, where v_i generates the kernel of δ_i. Let $\tilde{A} = \delta_i^*\delta_{i*}\delta_j^*\delta_{j*}A$ and similarly for $\tilde{B}$. The cycle $\tilde{A}$ (respectively, $\tilde{B}$) is well-defined independent of the order of δ_i, δ_j since it is obtained from $A \cap \mathcal{C}$ (respectively, $B \cap \mathcal{C}$) by adding uniquely weighted facets to all codimension one faces E of A (respectively, B), parallel only to the cones spanned by E and v_i for $i = 1, 2$, so that the result satisfies the balancing condition. Similarly, in $\mathbb{R}^k$ we have $\delta_{i*}\tilde{A} = \delta_{j*}\tilde{A}$ and analogously for $\tilde{B}$, since the weighted complexes $\delta_{i*}\tilde{A} \cap \delta(\mathcal{C})$ are equal for $i = 1, 2$ and balanced in all but the $\delta_i(v_j)$ direction where $j = 1, 2$ and $i \neq j$. Adding the necessary uniquely weighted facets to the codimension one faces of this complex in the $\delta_i(v_j)$ direction gives $\delta_{i*}\tilde{A}$ for $i = 1, 2$ and similarly for $\delta_{i*}\tilde{B}$. Also by construction we have $\delta_i^*\delta_{i*}\tilde{A} = \tilde{A}$, and similarly for B.

For $i=1,2$, define $\Delta^i_A = \delta_i^*\delta_{i*}A - A$ and $D^i_A = \text{div}_A(f_i)\times\mathbb{R} \subset D_i\times\mathbb{R}$ and similarly for B. Assume first that $A = \tilde{A} - \Delta^1_A - \Delta^2_A$, and analogously for B. It follows that $\delta_j^*\delta_{j*}\Delta^i_A = \Delta^i_A$, and similarly for B. Then we obtain,

$$A.B = \delta_i^*(\delta_{i*}(\tilde{A}-\Delta^j_A).(\delta_{i*}(\tilde{B}-\Delta^j_B)) - D^i_A.\Delta^i_B - \Delta^i_A.D^i_B + \Delta^i_A.\Delta^i_B$$

By distributivity, Lemma 3.7 and the assumption that $\delta_j^*\delta_{j*}\Delta^i_A = \Delta^i_A$ and $\delta_j^*\delta_{j*}\Delta^i_B = \Delta^i_B$ we have,

$$\begin{aligned} A.B = {} & \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B}) - \tilde{A}.\Delta^j_B - \Delta^j_A.\tilde{B} + \Delta^j_A.\Delta^j_B \\ & - D^i_A.\Delta^i_B - \Delta^i_A.D^i_B + \Delta^i_A.\Delta^i_B. \end{aligned}$$

The last three terms are products in $D_i\times\mathbb{R}$, and $\tilde{A}.\Delta^j_B, \Delta^j_A.\tilde{B}$ and $\Delta^j_A.\Delta^j_B$ are products in V. By applying the contraction δ_j to calculate these three products we obtain:

$$\Delta^j_A.\Delta^j_B - \tilde{A}.\Delta^j_B - \Delta^j_A.\tilde{B} = \Delta^j_A.\Delta^j_B - D^j_A.\Delta^j_B - \Delta^j_A.D^j_B.$$

Combining this with the equation above and we get,

$$\begin{aligned} A.B = {} & \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B}) - D^j_A.\Delta^j_B - \Delta^j_A.D^j_B + \Delta^j_A.\Delta^j_B \\ & - D^i_A.\Delta^i_B - \Delta^i_A.D^i_B + \Delta^i_A.\Delta^i_B, \end{aligned}$$

which is symmetric in i and j except for the first term $\delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B})$ which was already shown to be the same for $i=1,2$. So $A.B$ is independent of the contraction chart chosen.

Dropping our previous assumption, for any cycle we may still write $A = \tilde{A} - \Delta^1_A - \Delta^2_A - \Xi_A$, where Ξ_A is a cycle contained in the kernel of both δ_{1*} and δ_{2*}. Letting $A' = A + \Xi_A$, and analogously for B, and using distributivity with respect to either contraction chart we have

$$A.B = A'.B' - A'.\Xi_B - \Xi_A.B' + \Xi_A.\Xi_B. \tag{4}$$

As seen above, the product $A'.B'$ does not depend on the choice of chart δ_{i*}. Moreover since Ξ_A, Ξ_B are in the kernels of both δ_{i*} for both $i=1,2$, the product Ξ_A, Ξ_B descends to $D_{ij}\times\mathbb{R}^2$ where D_{ij} is the matroid corresponding to $M/\{i,j\}$ where M is the matroid of V. This doesn't depend on the order of i and j, see Section 3.1 of [18]. The other two products also descend to $D_{ij}\times\mathbb{R}^2$ as:

$$A'.\Xi_B = (\tilde{A} - \Xi_A).\Xi_B = (D^i_{\tilde{A}} + D^j_{\tilde{B}} - \Xi_A).\Xi_B$$
$$\Xi_A.B' = \Xi_A.(\tilde{B} - \Xi_B) = \Xi_A.(D^i_{\tilde{B}} + D^j_{\tilde{B}} - \Xi_B)$$

which are symmetric in i and j.

Now we treat the case of two elementary contractions. Let $\delta : V \longrightarrow V'$ be the composition of two elementary open matroidal contractions. First we set up notation to distinguish between the two orderings, similar to the proof of Proposition 2.29. We will call $\delta_i : V \longrightarrow V_i$ and $\tilde{\delta}_i : V_j \longrightarrow V'$ for $i \neq j$. Let $D_i \subset V_i$ be the divisor associated to δ_i and suppose $D_i = \text{div}_{V_i}(f_i)$. Similarly, $\tilde{D}_i \subset V'$ will denote the divisor of $\tilde{\delta}_i$ and $\tilde{f}_i$ its function. Keeping the notation from the beginning of the proof for Δ^i_A and D^i_A, we also set:

$$\begin{aligned} \tilde{\Delta}^i_A = {} & \tilde{\delta}_i^*\tilde{\delta}_{i*}A - A \subset V_j \\ \tilde{D}^i_A = {} & \text{div}_{\delta_* A}(\tilde{f}_i)\times\mathbb{R} \subset \tilde{D}_i\times\mathbb{R} \end{aligned}$$

Applying Definition 3.6 first by contracting with δ_i and then contracting with $\tilde{\delta}_j$ we obtain:

$$A.B = \delta^*(\delta_*A.\delta_*B) + C_i + \delta_j^*\tilde{C}_j$$

Where

$$C_i = \Delta^i_A.\Delta^i_B - \Delta^i_A.D^i_B - D^i_A.\Delta^i_B$$

with these three products calculated in $D_i\times\mathbb{R}$, and

$$\tilde{C}_j = \tilde{\Delta}^j_{\delta_{i*}A}.\tilde{\Delta}^j_{\delta_{i*}B} - \tilde{\Delta}^j_{\delta_{i*}A}.\tilde{D}^j_{\delta_{i*}B} - \tilde{D}^j_{\delta_{i*}A}.\tilde{\Delta}^j_{\delta_{i*}B}$$

with each product being calculated in $\tilde{D}_j \times \mathbb{R}$.

Again we first assume that $A = \delta^*\delta_*A + \Delta^1_A + \Delta^2_A$. Then we have $\delta_i^*\tilde{\Delta}^j_{\delta_{i*}A} = \Delta^j_A$. Restricting δ_j to $D_i \times \mathbb{R}$ we get an elementary open matroidal modification $\delta_j : D_i \times \mathbb{R} \longrightarrow \tilde{D}_i \times \mathbb{R}$. This can be checked on the level of the corresponding matroids. The divisor $\tilde{D}_i$ corresponds to the matroid $M\backslash j/i$ and contracting D_i by δ_j corresponds to $M/i\backslash j$. By Proposition 3.1.26 of [18] these matroids are equal. Now applying Lemma 3.7 for the products in $\tilde{D}_i \times \mathbb{R}$ we have, $C_i = \delta_i^*\tilde{C}_i$ and we obtain the same cycle regardless of order.

The general case follows an argument similar to the general case of two distinct elementary contractions to $\mathbb{R}^k$. We can once again write $A = \delta^*\delta_*A - \Delta^1_A - \Delta^2_A - \Xi_A$ and similarly for B. The rest of the argument follows exactly as above with the products in the end being in $D_{ij} \times \mathbb{R}^2$, where again D_{ij} corresponds to the matroid $M/\{i,j\}$. □

Now for weakly transverse intersections in a k-dimensional matroidal fan V we can make use of the definition of stable intersection in $\mathbb{R}^k$. For each facet F of V we can find a contraction chart $\delta : V \longrightarrow \mathbb{R}^k$ which does not collapse the face F. Recall, each facet of V corresponds to a maximal chain in the lattice of flats of the corresponding matroid. If after deleting an element i from the matroid the chain corresponding to F is still of length $k+1$, the tropical contraction δ_i of the Bergman fan does not collapse the face F. If the chain is of length $k+1$ on $n+1$ elements we can find $n-k$ elements to delete and not collapse F. Using this contraction chart to calculate the multiplicity we arrive at the following corollary.

Corollary 3.12. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and suppose the intersection of the two subcycles $A, B \subset V$ is weakly transverse when restricted to an open facet $F \subset V$ then $A.B \cap F$ corresponds to the stable intersection of Definition 2.4.*

Proposition 3.13. *For two cycles A, B in a matroidal fan $V \subset \mathbb{R}^n$, the product $A.B$ is supported on $(A \cap B)^{(m)}$, where m is the expected dimension of intersection.*

Proof. Once again our proof goes by induction. Given a facet F of $A.B$, choose a elementary open matroidal contraction chart $\delta : V \longrightarrow V'$ which does not contract the face $E \subset V$ containing F. Again, we may take any chart which does not contract all of the facets adjacent to E. Let f be the function on V' giving the modification δ and D the corresponding divisor. Then F is contained in $\Gamma_{V'}$ the graph of f. Let $\Gamma_D \subset \Gamma_{V'}$ be the graph of f restricted to D. If $\delta(F) \not\subset D$ then $\delta(F)$ must be a facet of $\delta_*A.\delta_*B$. By induction $\delta(F) \subset (\delta_*A \cap \delta_*B)^{(m)}$, so we must have $F \subset (A \cap B)^{(m)}$.

If on the other hand $F \subset \Gamma_D$ then $\delta(F)$ is an m dimensional face contained in $\delta_*A \cap \delta_*B \cap D$ where D the divisor of the elementary modification δ and F must be in one of the products $\Delta_A.D_B$, $\Delta_B.D_A$ or $\Delta_A.\Delta_B$ which occur in the fan $D \times \mathbb{R}$. Then assuming the statement holds on $D \times \mathbb{R}$, the facet F must be in one of $(\Delta_A \cap D_B)^{(m)} \cap \Gamma_D$, $(D_A \cap \Delta_B)^{(m)} \cap \Gamma_D$, or $(\Delta_A \cap \Delta_B)^{(m)} \cap \Gamma_D$. In any of these three cases F must be a facet of $(\Gamma_{\delta_*A} \cap \Gamma_{\delta_*B})^{(m)} \cap \Gamma_D$, and so in $(A \cap B)^{(m)}$. □

4. Two dimensional matroidal fans

In this section we consider $V \subset \mathbb{R}^n$ a two dimensional matroidal fan, and $A, B \subset V$ one dimensional fan tropical cycles. This means the vertex of A is the vertex of V, and similarly for B. Firstly, we simplify the definition of the intersection product given in the last section in this case.

For a two dimensional matroidal fan $V \subset \mathbb{R}^n$ we will consider the **coarse** subdivision on V described in general by Ardilia and Klivans in [3]. Suppose V corresponds to a matroid M which is loopless and contains no double points. A **double point** is an element $i \in E$ such that $r_M(\{i,j\}) = 1$ for some $j \in E$. Recall we defined the **fine** subdivision on V as the polyhedral complex $B(M)$ described in Section 2.4. When V is of dimension two and the corresponding matroid satisfies the above assumptions, the **coarse** subdivision of V is obtained from $B(M)$ by removing one-dimensional cones corresponding to flats M which are of rank two and size two or of size one

and contained in exactly two flats of rank two. See [3] for more details on the fine and coarse subdivisions of V.

Let $\delta : V \longrightarrow V'$, be an elementary open matroidal modification, f be the associated function on V' and D its divisor. The cycle Δ_A as defined in the last section is also a fan cycle and thus it is a union of rays. For any ray $\sigma_i \subset \Delta_A$ contained in the interior of a facet P_l of the coarse subdivision of V we may write $\sigma_i =< p_i v_1^l + q_i v_2^l >$, with $p_i, q_i \in \mathbb{N}$ with $(p_i, q_i) = 1$, where v_1^l, v_2^l are the primitive integer vectors corresponding to flats of M and spanning the one dimensional faces bounding P_l. Call A_{σ_i} the 1-cycle with three rays each in the directions of σ_i, v_1^l and v_2^l with weights $-w_A(\sigma_i)$, $w_A(\sigma_i)p_i$, and $w_A(\sigma_i)q_i$ respectively. Then the cycle A_{σ_i} is contained in the closure of P_l. Summing over the facets we get:

$$\Delta_A = \sum_{l=1}^{k} \sum_{\sigma_i \subset P_l^o} A_{\sigma_i}.$$

Given another fan subcycle $B \subset V$ we have an analogous decomposition

$$\Delta_B = \sum_{l=1}^{k} \sum_{\tau_j \subset P_l^o} B_{\tau_j}$$

where $\tau_j =< r_j v_1^l + s_j v_2^l >$ when $\tau_j \subset P_l^o$ and the cycles $B_{\tau_j} \subset P_l^o$ consist of the three rays τ_j, v_1^l and v_2^l with weights $-w_A(\tau_j)$, $w_A(\tau_j)r_j$, and $w_A(\tau_j)s_j$ respectively.

Since all of A, B, D are fans, the cycles D_A and D_B (as defined in the last section), are supported on the one skeleton of $D \times \mathbb{R}$. Take any contraction of $D \times \mathbb{R}$ which preserves the face F_l containing A_{σ_i}. Then D_B gets sent to an affine line in $\mathbb{R}^2$ with A_{σ_i} contained in a halfplane, so $A_{\sigma_i}.D_B = 0$ similarly with the roles of A and B interchanged. Moreover, if σ_i and τ_j are in different facets then $A_{\sigma_i}.B_{\tau_j} = 0$. The multiplicity of the vertex in $A.B$ becomes,

$$m_{A.B}(v) = m_{\delta_* A.\delta_* B}(\delta(v)) + m_{\Delta_A.\Delta_B}(v) = m_{\delta_* A.\delta_* B}(\delta(v)) + \sum_{l=1}^{k} \sum_{\sigma_i, \tau_j \subset P_l^o} A_{\sigma_i}.B_{\tau_j}.$$

The intersection of two cycles A_{σ_i}, B_{τ_j} in a face F_l can be calculated as stable intersection in $\mathbb{R}^2$ by Corollary 3.12 . If

$$\frac{s_j}{r_j} \geq \frac{p_i}{q_i}$$

then we can translate one of the two cycles so that they intersect in exactly one point of multiplicity $-w_A(\sigma_i)w_B(\tau_j)p_i r_j$. Otherwise,

$$\frac{p_i}{q_i} > \frac{s_j}{r_j}$$

and we can find a translation so that the two cycles intersect in exactly one point of multiplicity $-w_A(\sigma_i)w_B(\tau_j)s_i q_j$. We have just demonstrated the following proposition.

Proposition 4.1. *Let $V \subset \mathbb{R}^n$ be a two dimensional matroidal fan with vertex v and suppose $A, B \subset V$ are fan cycles and $v \in (A \cap B)^{(0)}$. Given a elementary contraction $\delta : V \longrightarrow V'$, and using the above notation, we have:*

$$m_{A.B}(v) = m_{\delta_* A.\delta_* B}(\delta(v)) - \sum_{l=1}^{k} \sum_{\sigma_i, \tau_j \subset P_l^o} w_A(\sigma_i)w_B(\tau_j) \min\{p_i r_j, q_i s_j\}.$$

Using this formula we prove the claim stated at the end of Example 2.28 by calculating the self intersection of $D \subset V$ where D is the divisor of any elementary open matroidal modification $\delta : \mathcal{M}_{0,n}^{trop} \longrightarrow V$. The fans resulting from each elementary open matroidal contraction of $\mathcal{M}_{0,5}^{trop}$ are

shown in Figure 4. The fan V is depicted by the graph in the upper right hand corner and $D \subset V$ is drawn in white. Performing another elementary contraction $\delta' : V \longrightarrow V'$ we obtain

$$m_{D.D}(v) = m_{\delta'_* D.\delta'_* D}(\delta'(v)) - 1.$$

Let $\delta'' : V' \longrightarrow \mathbb{R}^2$ be the composition of the last two contractions, in fact here we have $\delta''^* \delta''_* D = D$, so by Lemma 3.7 we have $m_{D.D}(v) = -1$. By part (5) of Proposition 3.9 $D.D = \mathrm{div}_D(f)$ so by Corollary 2.14 the function giving the elementary open matroidal modification $\delta : \mathcal{M}_{0,5} \longrightarrow V$ cannot be a regular function on $\mathbb{R}^4$.

In general, a one dimensional matroidal fan cycle in a two dimensional matroidal fan V corresponds to a matroid quotient of V. The next proposition describes the intersection multiplicity of two matroidal one cycles in V in terms of the lattices of flats of the corresponding matroids.

Theorem 4.2. *Let $V_M \subset \mathbb{R}^n$ be a two dimensional matroidal fan corresponding to the matroid M and $L_1, L_2 \subset V_M$ be two one dimensional tropical cycles corresponding to matroids M_1 and M_2 respectively. Then their intersection multiplicity at the vertex of V_M is*

$$m_{L_1.L_2}(v) = 1 - |\{F \mid r_M(F) = 2,\ F \in \Lambda(M_1) \cap \Lambda(M_2)\}|,$$

where $\Lambda(M_k)$ is the lattice of flats of M_k and r_M is the rank function on M.

Proof. To start, note that by a verification of the possible lines in $\mathbb{R}^2$ the theorem holds for $V_M = \mathbb{R}^2$. Given a two dimensional matroidal fan $V_M \subset \mathbb{R}^n$, let $\delta : V_M \longrightarrow V_{M\backslash i}$ denote a principal open matroidal modification. We may assume by induction that the given formula for the intersection multiplicity holds for $\delta_* L_1 = L'_1$ and $\delta_* L_2 = L'_2$ in $V_{M\backslash i}$. Now L'_k corresponds to the matroid $M_k \backslash i$ for $k = 1, 2$. Letting $\delta(v) = v'$ we obtain:

$$m_{L'_1.L'_2}(v') = 1 - |\{F \mid r_{M\backslash i}(F) = 2,\ F \in \Lambda(M_1\backslash i) \cap \Lambda(M_2\backslash i)\}|, \tag{5}$$

and $m_{\delta^*\delta_* L_1.\delta^*\delta_* L_2}(v) = m_{L'_1.L'_2}(v')$.

A ray σ_F of L_k corresponding to a flat $F \in \Lambda(M_k)\backslash\{\emptyset, E\}$ is contained in the interior of a facet of the undergraph $\mathcal{U}_f(V_{M\backslash i})$ considered with the coarse subdivision if and only if the corresponding flat F is of rank two in M and contains i. The weights of all edges of L_1, L_2 are equal to one, so in this situation the simplification given in Proposition 4.1 yields,

$$m_{L_1.L_2}(v) = m_{L'_1.L'_2}(v') - |\{F \mid r_M(F) = 2,\ F \in \Lambda(M_1) \cap \Lambda(M_2),\ i \in F\}. \tag{6}$$

The combination of Equations 5 and 6 along with the description of the flats of $M\backslash i$ and M/i given in the proof of Proposition 2.22 proves the intersection multiplicity for L_1 and L_2 in V_M. ☐

It is possible generalise the above proposition to a matroidal fan V_M of any dimension and describe combinatorially the product of two fans corresponding to matroidal quotients $V_1, V_2 \subset V_M$. This product on matroidal quotients generalises the standard matroid intersection, which is shown by Speyer to correspond to tropical stable intersection under certain conditions, [22].

Example 4.3. *Denote the Bergman fans of the Fano plane and the anti-Fano by V_F and V_{F^-} respectively. Both are contained in $\mathbb{R}^6$, and they can both be contracted to $\mathcal{M}^{trop}_{0,5} \subset \mathbb{R}^5$ via an elementary open matroidal contraction, see Example 1.5.6 of* [18]. *Let $D_F, D_{F^-} \subset \mathcal{M}^{trop}_{0,5}$ denote the divisors of the contractions of F and F^- respectively. Then D_F is a tropical one cycle with three rays and D_{F^-} is a tropical one cycle with four rays. The cycles D_F and D_{F^-} have two rays in common. Moreover, these common rays correspond to flats of rank two in the matroid corresponding to $\mathcal{M}^{trop}_{0,5}$, therefore by Theorem 4.2 we obtain,*

$$m_{D_F.D_{F^-}}(v) = -1.$$

A negative intersection multiplicity of two effective subcycles is under some circumstances an indication that these cycles cannot both arise as tropicalisations of classical varieties. The following theorem makes this precise and is due to an observation of E. Brugallé. Here, let $\mathbf{K}$ be the field of Puiseux series with coefficients in an algebraically closed field $\mathbf{k}$, and let $\text{Trop}(\mathbf{V}) \subset \mathbb{R}^n$ denote the tropicalisation of a subvariety $\mathbf{V} \subset (\mathbf{K}^*)^n$ from [12]. We say that a subvariety $\mathbf{V} \subset (\mathbf{K}^*)^n$ is a plane if it is two dimensional and defined by a system of linear equations. Let $\tilde{\mathbf{V}} \subset (\mathbf{K}^*)^n$ be a plane and $\Delta : (\mathbf{K}^*)^n \longrightarrow (\mathbf{K}^*)^{n-1}$ be the projection by forgetting a coordinate direction, then $\Delta(\tilde{\mathbf{V}}) = \mathbf{V}$ is also a plane. Let $\text{Trop}(\tilde{\mathbf{V}}) = \tilde{V}$ and $\text{Trop}(\mathbf{V}) = V$, then there is a tropical modification $\delta : \tilde{V} \longrightarrow V$. Denote its corresponding divisor $D \subset V$. Using this notation we have the following theorem.

Theorem 4.4. *Let $\tilde{\boldsymbol{V}} \subset (\boldsymbol{K}^*)^n$ be a plane and let $\boldsymbol{V} \subset (\boldsymbol{K}^*)^{n-1}$ be the projection of $\tilde{\boldsymbol{V}}$ along one of the coordinate directions. Suppose $Trop(\tilde{\boldsymbol{V}}) = \tilde{V} \subset \mathbb{R}^n$ and $Trop(\boldsymbol{V}) = V \subset \mathbb{R}^{n-1}$ and let $\delta : \tilde{V} \longrightarrow V$ be an open elementary tropical modification with divisor $D \subset V$. Given a tropical curve $C \subset V$ such that $D \not\subset C$, if there exists a bounded connected subset $Q \subset C \cap D$ such that*

$$\sum_{p \in Q \cap (D \cap C)^{(0)}} m_p(D.C) < 0$$

then there is no algebraic curve $\boldsymbol{C} \subset \boldsymbol{V}$ such that $Trop(\boldsymbol{C}) = C$.

Proof. The plane $\tilde{\mathbf{V}} \subset (\mathbf{K}^*)^n$ is obtained by taking the graph of a linear function $\mathbf{f}$ on $\mathbf{V}\backslash\mathbf{D}$ where $\mathbf{D}$ is the divisor of $\mathbf{f}$ on $\mathbf{V}$. Suppose there exists a $\mathbf{C} \subset \mathbf{V}$ such that $\text{Trop}(\mathbf{C}) = C$. Then $\mathbf{f}$ also gives an embedding $\mathbf{C}\backslash\mathbf{C}\cap\mathbf{D} \longrightarrow \tilde{\mathbf{V}} \subset (\mathbf{K}^*)^n$ given by taking the graph of $\mathbf{f}$ restricted to $\mathbf{C}\backslash\mathbf{C}\cap\mathbf{D}$. Let $\tilde{\mathbf{C}}$ denote the image of this embedding and let $\tilde{C} = \text{Trop}(\tilde{\mathbf{C}})$. By Theorem 3.3.4 of [12] there exist positive weights on the facets of $\tilde{C}$ making it a balanced cycle. However, the pullback δ^*C is not-effective, in particular for each point $p \in Q$ with $m_p(D.C) \neq 0$ there is a corresponding half-ray in δ^*C in the direction $-e_n$ of weight $m_p(D.C)$, whose image under δ is the point p. The cycles $\tilde{C}$ and δ^*C agree as weighted complexes outside of $\delta^{-1}(D \cap C)$. Moreover, the difference $\tilde{C} - \delta^*C$ is a cycle and has a connected component contained in $\delta^{-1}(Q)$. Since Q is bounded, all of the unbounded rays of $\tilde{C} - \delta^*C$ in $\delta^{-1}(Q)$ must have primitive integer direction $-e_n$. The recession fan of $\tilde{C} - \delta^*C$ is also balanced, meaning the sum of the weights of the unbounded edges of $\tilde{C} - \delta^*C$ must also be equal zero. However, the sum of the weights of the unbounded edges of δ^*C is given by

$$\sum_{p \in Q \cap (D \cap C)^{(0)}} m_p(D.C) < 0,$$

therefore the cycle $\tilde{C}$ may not be effective. This contradiction proves the theorem. ☐

Proposition 4.5. *Recall the curve B from Example 3.2. For $d \geq 3$, $B \subset V$ is not realisable over any field.*

Proof. By the above theorem it suffices to show that the matroid corresponding to $\tilde{V}$ which is the fan obtained by the modification $\delta : \tilde{V} \longrightarrow V$ along the matroidal divisor A is a regular matroid, i.e. realisable over every field. For a matroid of this rank on only five elements we must only check that it has no minors corresponding to the four point line, see Theorem 6.6.4 of [18]. Tropically this means that the divisor of any contraction cannot be the four valent tropical line $L \subset \mathbb{R}^3$. Verifying the five possible contractions and we see that it holds. ☐

This is a light version of a much stronger result which should hold not just in open Bergman fans but in their compactifications as well and in non-singular tropical varieties.

Unfortunately, there are some tropical 1-cycles which are not realisable which pass this *intersection test*. For example, Vigeland's 1-parameter family of lines on a degree $d \geq 3$ surface, see Theorem 9.3 of [24].

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